COSMOLOGY - COURSE @ T8

DYNAMICS OF STRUCTURE FORMATION I

BY KATRIN

OUTLINE :

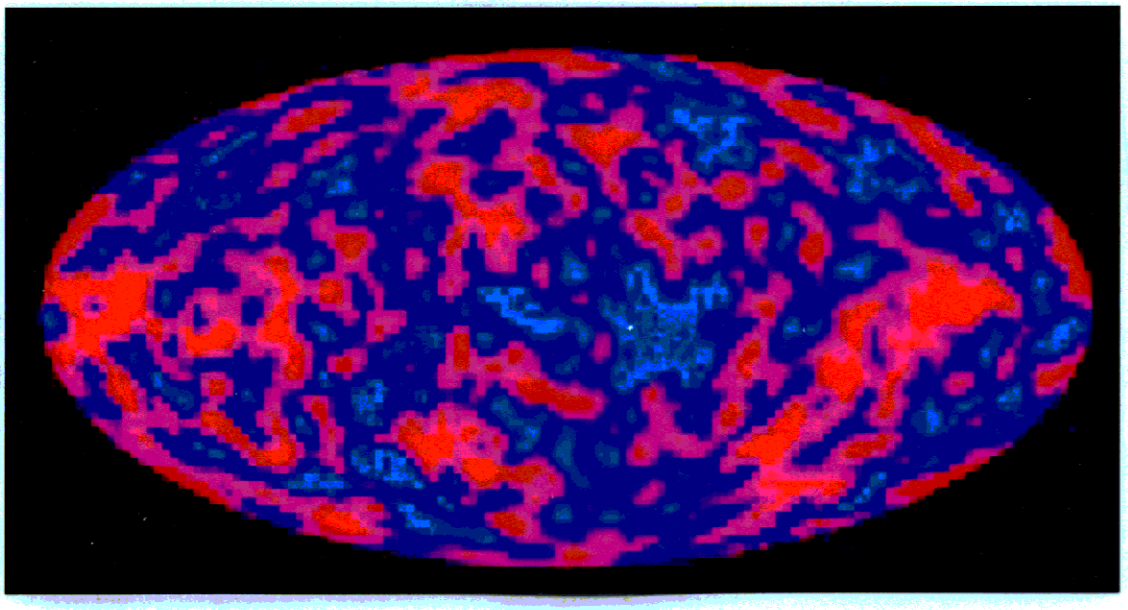
I. INTRODUCTION

II. JEANS THEORY - GRAVITATIONAL INSTABILITY

III. DYNAMICS OF LINEAR PERTURBATIONS IN EXPANDING UNIVERSE

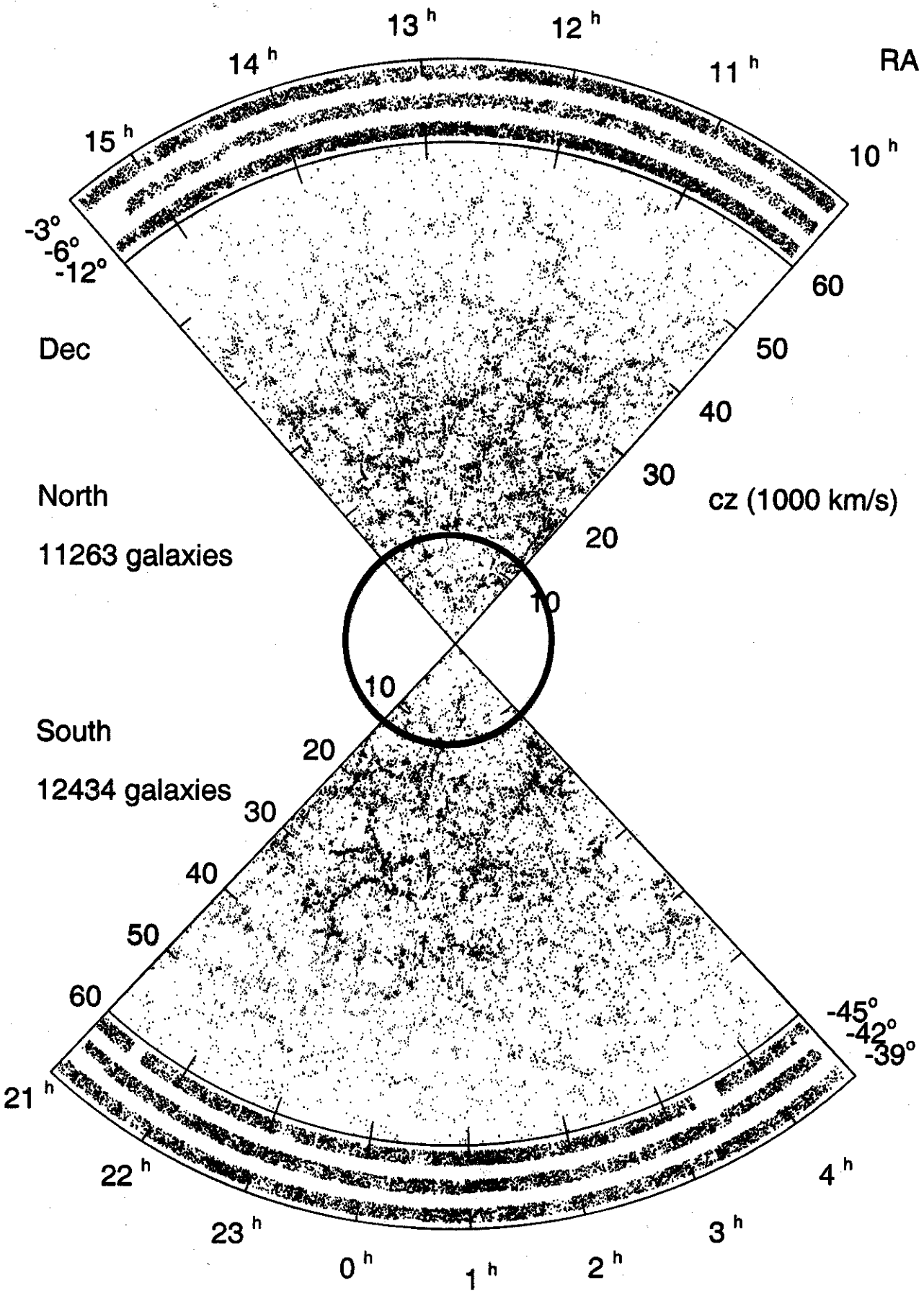
IV. CONCLUSIONS

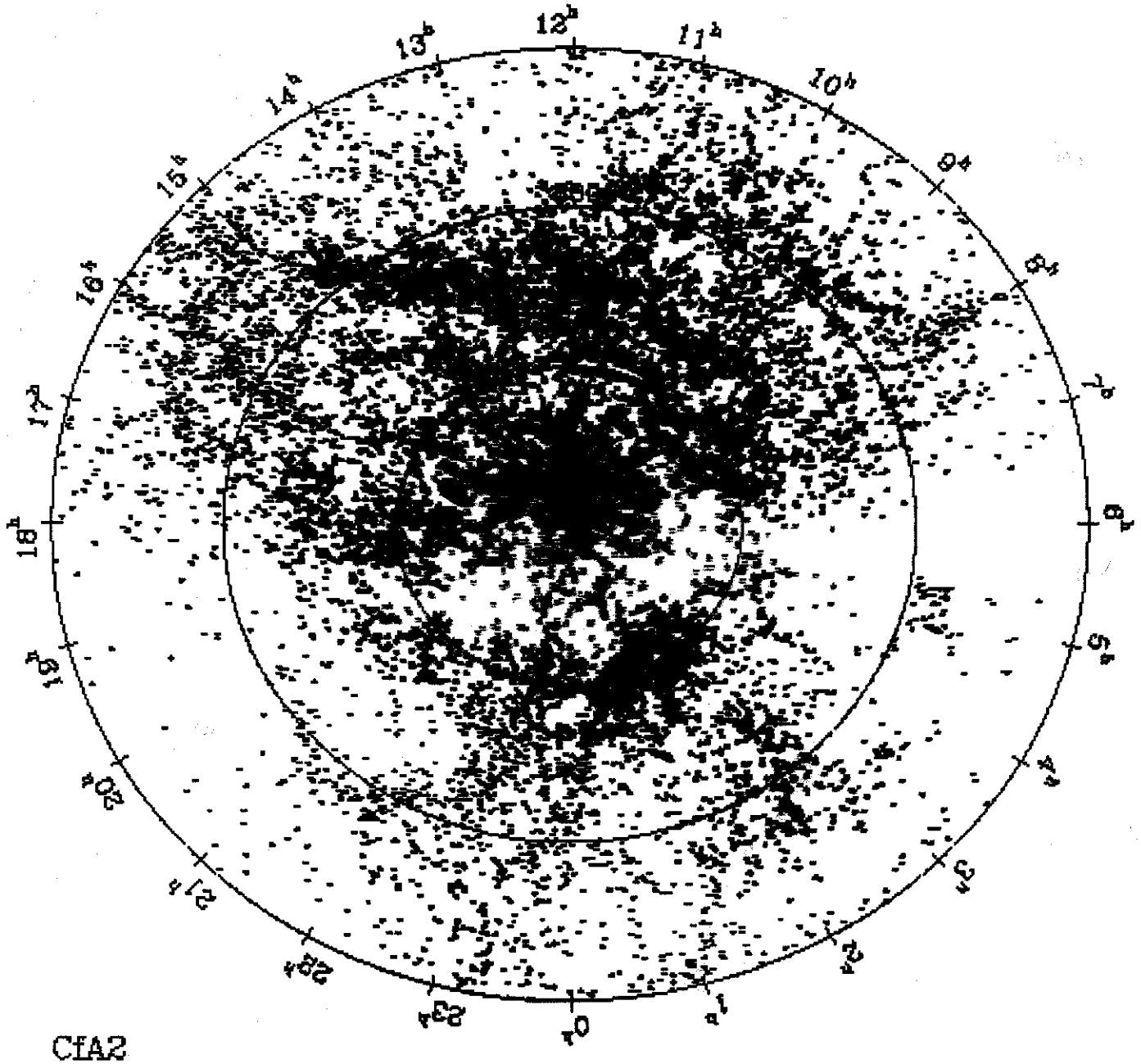
- COsmic Background Explorer, 1992



Sky map of the background radiation

scale: -100 μK to 100 μK


13 h
12 h
14 h
11 h
RA
15 h
10 h
-3°
-6°
-12°
Dec
60
50
40
30
North
cz (1000 km/s)
11263 galaxies
20
10
10
South
20
12434 galaxies
30
40
50
60
-45°
-42°
-39°
21 h
4 h
22 h
3 h
23 h
0 h
1 h
2 h



CfA2

Max Radius 12000
0 ≤ h < 12000 (km/s)
m_B ≤ 15.5

Puck

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I. INTRODUCTION

- REAL UNIVERSE CONTAINS INHOMOGENEOUS STRUCTURES: GALAXIES, CLUSTERS ...
- WHERE DO INHOMOGENEITIES COME FROM ?
- HOW CAN WE DESCRIBE A MECHANISM FOR FORMATION
- ASSUME: SOME TIME IN THE PAST: SMALL DEVIATIONS FROM HOMOGENEITY

DEVIATIONS GROW DUE TO GRAVITATIONAL INSTABILI

EVENTUALLY: FORM GALAXIES, CLUSTERS ...

- AS LONG AS INHOMOGENEITIES ARE SMALL:

  → LINEAR PERTURBATION THEORY

- IF NOT: NONLINEAR EVOLUTION !

NONLINEAR PROCESSES

- MODERN COSMOLOGIST:
- CONFRONT PREDICTIONS ↔ OBSERVATIONS
- ASTRONOMER SEES:
- RESULT OF NONLINEAR GRAVITATIONAL EVOLUTION
- CLUSTERS OF GALAXIES THOUSANDS OF TIMES DENSER THAN MEAN
- DENSITIES OF GALAXIES MILLIONS TIMES THE AVERAGE

⇒ UNDERSTANDING OF PROCESSES THAT MODIFY SIMPLE LINEAR EVOLUTION GROWTH LAWS IS VERY IMPORTANT !

- BRUTE FORCE : NUMERICS LARGE # OF PARTICLES, GIVE SMALL INITIAL PERTURBATIONS

- HERE : LINEAR PERTURBATION THEORY

- APPROACH ASSUMES EXISTENCE OF SMALL INHOMOGENEITIES AT SOME INITIAL TIME

- COMPLETE MODEL SHOULD ALSO PRODUCE THESE INHOMOGENEITIES

⇒ LATER IN COSMO @ T8 !!

II. JEANS' THEORY - GRAVITATIONAL INSTABILITY

- BEGINNING OF LAST CENTURY:

  JEANS INVESTIGATES FORMATION OF STARS

- FINDS GRAVITATIONAL JEANS INSTABILITY

- CORNERSTONE OF MODEL FOR ORIGIN OF GALAXIES AND LARGE SCALE STRUCTURES

- START: HOMOGENEOUS, ISOTROPIC FLUID

  => SMALL FLUCTUATIONS IN DENSITY $\delta\rho$,

  VELOCITY $\delta\vec{v}$, CAN EVOLVE IN TIME

- STATIC BACKGROUND, NO EXPANSION

=> $\delta\rho$ GROWS WITH TIME IF STABILISING

EFFECT OF p << TENDENCY OF SELFGRAVITY

OF $\delta\rho$ TO INDUCE COLLAPSE

- EASY TO UNDERSTAND:
  - GRAVITY IS ATTRACTIVE FORCE
  - AS LONG AS ρ IS SMALL: OVERDENSE REGION ACCRETES MATERIAL FROM SURROUNDING
  - BECOMES MORE DENSE
  - THE DENSER, THE MORE ACCRETES

=> COLLAPSE OF FLUCTUATION TO GRAVITATIONALLY BOUND OBJECT !

- CRITERION FOR GROWTH OF FLUCTUATIONS:

  LENGTHSCALE OF FLUCTUATION > λ_J

  - WHAT IS λ_J : JEANSWAVELENGTH ?
  - CONSIDER: DYNAMICS OF SELFGRAVITATING FLUID STATIC BACKGROUND

- NOW: PERTURB EQUATIONS:

$$\rho = \rho_0 + \delta\rho \qquad \vec{v} = \vec{v}_0 + \delta\vec{v}$$

$$p = p_0 + \delta p_0 \qquad s = s_0 + \delta s$$

$$\varphi = \varphi_0 + \delta\varphi_0$$

- INSERT, ONLY LINEAR PERTURBATIONS SURVIVE:

$$\delta\dot{\rho} + \rho_0 \nabla \cdot \delta\vec{v} = 0$$

$$\delta\dot{\vec{v}} + \frac{1}{\rho_0}\left(\frac{\partial p}{\partial \rho}\right)_s \nabla\delta\rho + \frac{1}{\rho_0}\left(\frac{\partial p}{\partial s}\right)_\rho \nabla\delta s + \nabla\delta\varphi = 0$$

$$\nabla^2\delta\varphi - 4\pi G\,\delta\rho = 0$$

$$\delta\dot{s} = 0$$

- NOW: SOLVE THESE EQUATIONS

- ANSATZ: PLANE WAVES

$$\delta u_i = \delta_i\, e^{i\vec{k}\cdot\vec{r}} \qquad \delta u_i = \delta\rho, \delta\vec{v}, \delta\varphi, \delta s$$

$$\delta_i(t) = \delta_{0i}\, e^{i\omega t} \qquad \delta_{0i} = \mathcal{D}, \vec{\mathcal{V}}, \Phi, \Sigma$$

• PLUG IN ...

(1) $\dot{\delta\rho} + \rho_0 \nabla \cdot \delta\vec{v} = 0 \Rightarrow \omega D + \rho_0 \vec{k} \cdot \vec{V} = 0$

$\Rightarrow \omega \delta_0 + \vec{k} \cdot \vec{V} = 0 \qquad \delta_0 \equiv \frac{D}{\rho_0}$

(2) $\dot{\delta\vec{v}} + \frac{1}{\rho_0} \underbrace{\left(\frac{\partial p}{\partial \rho}\right)_s}_{= v_s^2} \nabla \delta\rho + \frac{1}{\rho_0} \left(\frac{\partial p}{\partial s}\right)_\rho \nabla \delta s + \nabla \delta\varphi = 0$

$\Rightarrow \omega \vec{V} + v_s^2 \vec{k} \delta_0 + \frac{\vec{k}}{\rho_0} \left(\frac{\partial p}{\partial s}\right)_\rho \Sigma + \vec{k} \Xi = 0$

(3) $\nabla^2 \delta\varphi - 4\pi G \delta\rho \Rightarrow k^2 \Xi + 4\pi G \rho_0 \delta_0 = 0$

(4) $\dot{\delta s} = 0 \Rightarrow \omega \Sigma = 0$

(1) $\omega_0 \delta_0 + \vec{k} \cdot \vec{V} = 0$

(2) $\omega \vec{V} + \vec{k} v_s^2 \delta_0 + \frac{\vec{k}}{\rho_0} \left(\frac{\partial p}{\partial s}\right)_\rho \Sigma + \vec{k} \Xi = 0$

(3) $k^2 \Xi + 4\pi G \rho_0 \delta_0 = 0$

(4) $\omega \Sigma = 0$

• CONSIDER TIME-DEPENDENT SOLUTION $\omega \neq 0$

(4) $\Rightarrow \Sigma = 0$, ADIABATIC PERTURBATIONS

(1) $\Rightarrow \vec{k} \cdot \vec{V} \neq 0$, RESOLVE INTO COMPONENTS

(a) $\vec{k} \perp \vec{V}$: VORTICAL MODES, $\nabla \times \vec{v} \neq 0$

NO PERTUBATIONS TO DENSITY

(b) $\vec{k} \parallel \vec{V}$: LONGITUDINAL MODES

$$\Rightarrow \quad \omega \delta_0 + kV = 0$$

$$\omega V + k v_s^2 \delta_0 + k \Phi = 0$$

$$k^2 \Phi + 4\pi G \rho_0 \delta_0 = 0$$

ONLY NON-ZERO SOLUTION, IF $\det = 0$

$\Rightarrow$ DISPERSION-RELATION: $\boxed{\omega^2 = v_s^2 k^2 - 4\pi G \rho_0}$

• COMPARE WITH: DISPERSION-RELATION FOR LONGITUDINAL ELECTROSTATIC OSCILLATIONS IN PLASMA

$$\omega^2 = v_s^2 k^2 - 4\pi G \rho_0$$

$$\omega^2 = v_s^2 k^2 + 4\pi m_e e^2/m_e$$

- $n_e \hat{=} \rho_0/m$
- $m_e \hat{=} m$
- $e^2 \hat{=} G m^2$
- MINUS: ATTRACTIVE NATURE OF GRAVITATION

- MINUS: "GRAVITOSTATIC" WAVES HAVE INSTABILITY

- TWO TYPES OF SOLUTION:

$$\lambda = 2\pi/k \Rightarrow \omega^2 = \frac{4\pi^2 v_s^2}{\lambda^2} - 4\pi G \rho_0 \overset{!}{=} 0$$

$$\Rightarrow \lambda = v_s \sqrt{\frac{\pi}{G\rho_0}} \equiv \lambda_J$$ JEANS-WAVELENG

- DISPERSION-RELATION IN TERMS OF λ_J:

$\lambda < \lambda_J$:

$$\omega = \pm v_s k \left[1 - \frac{4\pi G \rho_0}{v_s^2 k^2}\right]^{1/2} = \pm v_s k \left[1 - \left(\frac{\lambda}{\lambda_J}\right)^2\right]^{1/2}$$

$$\frac{4\pi G \rho_0}{v_s^2 k^2} = \frac{\pi}{\lambda_J^2}$$

- $\lambda < \lambda_J$: ω REAL

$\Rightarrow$

$$\frac{\delta\rho}{\rho_0} = \delta_0 \exp\left[i(\vec{k}\cdot\vec{r} \pm |\omega| t)\right]$$

$$\delta\vec{v} = \mp \frac{\vec{k}}{k} v_s \delta_0 \left[1 - \left(\frac{\lambda}{\lambda_J}\right)^2\right]^{1/2} \exp\left[i(\vec{k}\cdot\vec{r} \pm |\omega| t)\right]$$

$$\delta\varphi = -\delta_0 v_s^2 \left(\frac{\lambda}{\lambda_J}\right)^2 \exp\left[i(\vec{k}\cdot\vec{r} \pm |\omega| t)\right]$$

- TWO SOUNDWAVES IN DIRECTION $\pm\vec{k}$
- PHASE VELOCITY TENDS TO ZERO FOR $\lambda \to \lambda_J$
- $\lambda > \lambda_J$: ω COMPLEX

$$\omega = \pm i (4\pi G \rho_0)^{1/2} \left[1 - \frac{v_s^2 4\pi^2}{\lambda^2 4\pi G\rho_0}\right] = \pm i (4\pi G\rho_0)^{1/2} \left[1 - \left(\frac{\lambda_J}{\lambda}\right)^2\right]^{1/2}$$

($v_s^2 4\pi^2 = \lambda_J^2$, $4\pi G\rho_0$)

- CHARACTERISTIC TIME-SCALE:

$$\tau \equiv |\omega|^{-1} = (4\pi G\rho_0)^{-1/2} \left[1 - \left(\frac{\lambda_J}{\lambda}\right)^2\right]^{-1/2}$$

INSTABILITY !

τ DIVERGENT FOR $\lambda \to \lambda_J$

- $\lambda > \lambda_J$:

$$\frac{\delta\rho}{\rho_0} = \delta_0 \exp\left[i\vec{k}\cdot\vec{r} \pm i\omega t\right]$$

$$\delta\vec{v} = \mp i \frac{\vec{k}}{k} \delta_0 (4\pi G\rho_0)^{1/2} \left[1-\left(\frac{\lambda_J}{\lambda}\right)^2\right]^{1/2} \exp\left[i\vec{k}\cdot\vec{r} \pm i\omega t\right]$$

$$\delta\varphi = -\delta_0 v_s^2 \left(\frac{\lambda}{\lambda_J}\right)^2 \exp\left[i\vec{k}\cdot\vec{r} \pm i\omega t\right]$$

- NON-PROPAGATING SOLUTION (STATIONARY WAVE) WITH INCREASING OR DECREASING AMPLITUDE

- WHAT TO DO ABOUT INSTABILITY ?

! EXPANSION OF THE UNIVERSE !

III. DYNAMICS OF LINEAR PERTURBATIONS

YOU HAVE TO DEAL WITH LINEARIZED GR! BUT WATCH OUT AND DON'T SCREW UP! THINK ABOUT GAUGE INVARIANCE!

DON'T LISTEN TO HIM! ESSENTIAL PHYSICS CAN BE EXTRACTED OUT OF MY THEORY! REMEMBER: SMALL PERTURBATIONS IMPLY WEAK GRAVITATIONAL FIELDS!

WHO IS RIGHT ?

① EINSTEIN

- CONSIDER PERTURBATIONS OF

METRIC : $g_{\alpha\beta}(x) \rightarrow g_{\alpha\beta}(x) + \delta g_{\alpha\beta}(x)$

SOURCE : $T_{\alpha\beta}(x) \rightarrow T_{\alpha\beta}(x) + \delta T_{\alpha\beta}(x)$

→ LINEARIZED EINSTEIN'S EQUATIONS:

$$\hat{\mathcal{L}}(g_{\alpha\beta})\,\delta g_{\alpha\beta} = \delta T_{\alpha\beta}$$

$\hat{\mathcal{L}}$: LIN. DIFF. OPERATOR, DEPENDS ON BACKGROUND

- LINEAR EQUATION → EXPAND SOLUTIONS IN MODEFUNCTIONS
- $\Omega = 1$: MODEFUNCTIONS = PLANE WAVES
- FOURIER TRAFO : $\hat{\mathcal{L}}_{(k)}\,\delta g_{(k)} = \delta T_{(k)}$

- SOLVING THIS SET OF EQUATIONS

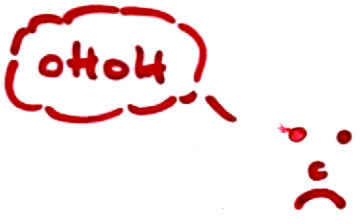
→ EVOLUTION FOR EACH MODE SEPARATELY

- BUT: IN GR, $g_{\alpha\beta}$ OR $T_{\alpha\beta}$ CAN BE CHANGED BY RELABELLING THE COORDINATES:

$$x^{\alpha} \to x^{\alpha'}$$

→ TRIVIAL CHANGE CAN MAKE A SMALL $\delta T_{\alpha\beta}$ LARGE OR GENERATE A NEW COMPONENT....

→ GAUGE AMBIGUITIES!



② NEWTON

- CAN HANDLE PROBLEMS FOR MODES WITH PROPER WAVELENGTH < HUBBLE RADIUS
- AT THIS LENGTH SCALES: GR-EFFECTS DUE TO CURVATURE OF SPACETIME NEGLIGIBLE
- BUT: AT VERY EARLY EPOCHS PROPER WAVELENGTH > HUBBLE RADIUS

⇒ NEWTON VALID FOR TIMES $t \gg t_{\text{ENTER}}(\lambda)$

THEN MODE WELL INSIDE HUBBLE RADIUS

⇒ EARLY EVOLUTION OF ANY MODE NEEDS TO BE TACKLED BY GR (...OR SALMAN...)

NEWTON WINS IN THIS TALK!

- START ONCE AGAIN ....

$$\frac{d}{dt}\delta\vec{v} = -\frac{\nabla\delta p}{\rho_0} - \nabla\delta\varphi - \boxed{(\delta\vec{v}\cdot\nabla)\vec{v}_0}$$

WE SAW THESE ALREADY !

$$\frac{d}{dt}\delta = -\nabla\cdot\delta\vec{v}$$

$$\nabla^2\varphi = 4\pi G\rho_0\delta$$

- WITH : $\delta \equiv \frac{\delta\rho}{\rho_0}$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{v}_0\cdot\nabla$$

= t-DERIVATIVE FOR θ, COMOVING WITH UNPERTURBED EXPANSION OF UNIVERSE

$$\square = [(\delta\vec{v}\cdot\nabla)\vec{v}_0]_j = [\delta v]_i \nabla_i [v_0]_j = H[\delta v]_j$$

$$\vec{v}_0 = H\vec{x}_0 \Rightarrow \nabla_i [v_0]_j = H\delta_{ij}$$

- EQUATIONS IN EUCLIDIAN COORDINATES

→ PROPER LENGTH UNITS

→ HUBBLE EXPANSION EXPLICT THROUGH $\vec{v}_0$

- ALTERNATIVE: LAGRANGIAN COORDINATES

  POSITION OF PARTICLE DOESN'T CHANGE WITH t

⇒ EULERIAN AT SOME FIXED t_0 = LANGRANGIAN C

- LAGRANGIAN COO. ≙ COMOVING COO., LABEL θ WHO

  FOLLOWS HUBBLE EXPANSION IN UNPERTURBED UNIVE

- ONCE CONSIDER PERTURBED UNIVERSE:

  EULER COORDINATES $/a(t) \neq$ LANGRANGIAN COO.

- GRAVITY CAUSES NON-UNIFORM DISTRIBUTION OF

  PARTICLES TO GROW INCREASINGLY IRREGULAR

⇒ MATTER DISPLACED W.R.T. UNIFORMLY EXPANDING

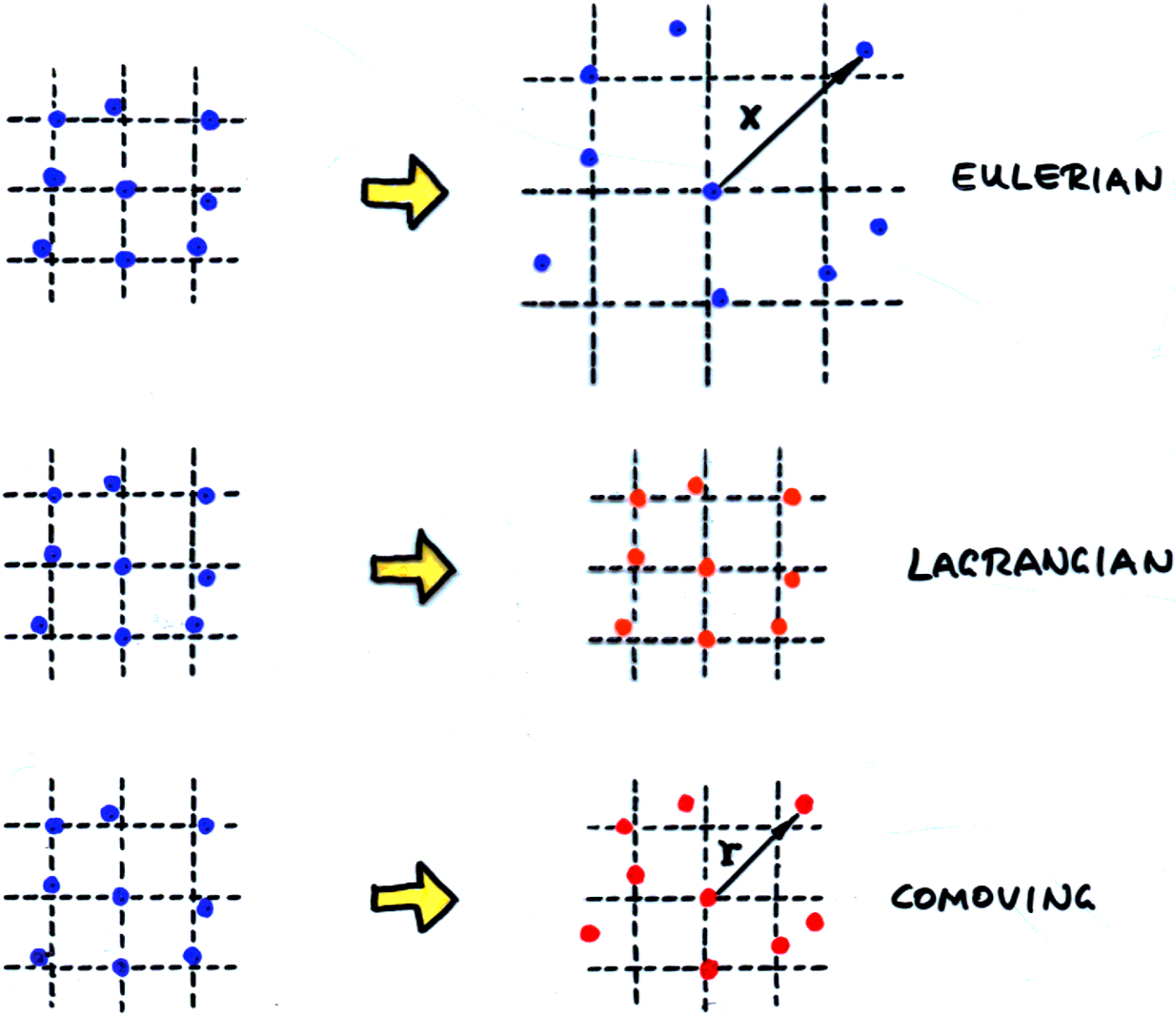
  BACKGROUND

- DISPLACEMENT MOST DIRECT SIGNATURE

  OF INHOMOGENEITY

  ⇒ COMOVING UNITS !

DIFFERENT COORDINATES



- DEFINE : COMOVING QUANTITIES

$$\vec{x}(t) = a(t)\,\vec{r}(t)$$

$$\delta\vec{v}(t) = a(t)\,\vec{u}(t)$$

$$\nabla_x = \frac{1}{a(t)}\nabla_r \equiv \frac{1}{a(t)}\nabla$$

- PLUG IN...

(1) $$\dot{\vec{u}} + 2\frac{\dot{a}}{a}\vec{u} = \frac{\vec{g}}{a} - \frac{\nabla\delta p}{\rho_0}$$

(2) $$\dot{\delta} = -\nabla\cdot\vec{u}$$

WITH: $\vec{g} = \nabla\delta\varphi/a$

- THREE EQUATIONS FOR FOUR VARIABLES
- NEED EQUATION OF STATE $v_s^2 \equiv \frac{\partial p}{\partial \rho}$
- THINK OF PLANE-WAVE DISTURBANCE

∇(1) → $\partial/\partial t$ (2)

→ $$\boxed{\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \delta\left(4\pi G\rho_0 - \frac{v_s^2 k^2}{a^2}\right)}$$

GRAVITATIONAL AMPLIFICATION OF PERTURBATIONS

RADIATION DOMINATED UNIVERSE

- EARLY ENOUGH TIMES: RADIATION DOMINATED

$v_s = c/\sqrt{3}$ ANALYSIS SO FAR:

- AT THIS POINT: EINSTEIN RIGHT

- BUT: FIELDS ARE STILL WEAK

- COMBINE: SPECIAL RELATIVITY FLUID DYNAMICS

  + NEWTONIAN GRAVITY

  + RELATIVISTIC SOURCE TERM

  + $\lambda \gg \lambda_J$

  + $p = \rho/3$

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \frac{8}{3}4\pi G\rho_0\delta - \frac{1}{3}\frac{k^2c^2}{a^2}\delta$$

SOLUTIONS FOR $\delta(t)$

- $\rho_0 \sim t^{-2}$ FOR MATTER AND RADIATION FOR $\Omega = 1$
- FOLLOWS FROM $\Omega = \frac{8\pi G \rho_0}{3H^2} = \frac{8\pi G \rho_0}{3}\left(\frac{a}{\dot{a}}\right)^2 = 1$
- MATTER: $a \sim t^{2/3} \Rightarrow \dot{a} = \frac{2}{3} t^{-1/3} \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{2}{3t}\right)^2$

  $\Rightarrow 4\pi G \rho_0 = \frac{2}{3t^2}$
- RADIATION: $a \sim t^{1/2} \Rightarrow \dot{a} = \frac{1}{2} t^{-\frac{1}{2}} \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{1}{2t}\right)^2$

  $\Rightarrow \frac{32}{3}\pi G \rho_0 = \frac{1}{t^2}$
- FOR $\lambda \gg \lambda_J$ (p NEGLIGIBLE COMPARED TO GRAVITY):

  $\frac{k^2 c_s^2}{a^2}\delta$, $\frac{1}{3}\frac{k^2 c^2}{a^2}\delta$ CAN BE IGNORED

$\Rightarrow$ EVERY TERM IN EQS. FOR δ IS

PROPORTIONAL TO $\delta, \dot{\delta}, \ddot{\delta}, t$

$\Rightarrow$ POWERLAW SOLUTION!

ANSATZ: $\delta \sim t^n$

- MATTER: $\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0$

  ⇒ GROWING MODE: $\boxed{\delta \sim t^{2/3} \sim a}$

  DECAYING MODE: $\delta \sim t^{-1}$

- RADIATION: $\ddot{\delta} + \frac{1}{t}\dot{\delta} - \frac{1}{t^2}\delta = 0$

  ⇒ GROWING MODE: $\boxed{\delta \sim t \sim a^2}$

  DECAYING MODE: $\delta \sim t^{-1}$

- COMBINE MATTER / RADIATION VIA η:

  $\eta = \int \frac{dt}{a}$: MATTER $\eta = \int \frac{dt}{t^{2/3}} = 3t^{1/3} \sim \delta^{1/2}$

  RADIATION $\eta = \int \frac{dt}{t^{1/2}} = 2t^{1/2} \sim \delta^{1/2}$

  ⇒ $\boxed{\eta^2 \sim \delta}$

IV. CONCLUSIONS

- SIMPLE JEANS-ANALYSIS LEADS TO INSTABILITY, EXPONENTIAL GROWTH OF FLUCTUATIONS

- TAKING EXPANSION INTO ACCOUNT POWERLAW GROWTH

- HERE: ONLY TOY-MODELS!

  WITH NEWTON: $t \gg t_{ENTER}(\lambda)$

  $d_H \gg \lambda \gg \lambda_J$

- MORE PHYSICS: NEXT WEAK !!

  TO NEXT