The Inflationary Perturbation Spectrum – Why the Early Universe Matters

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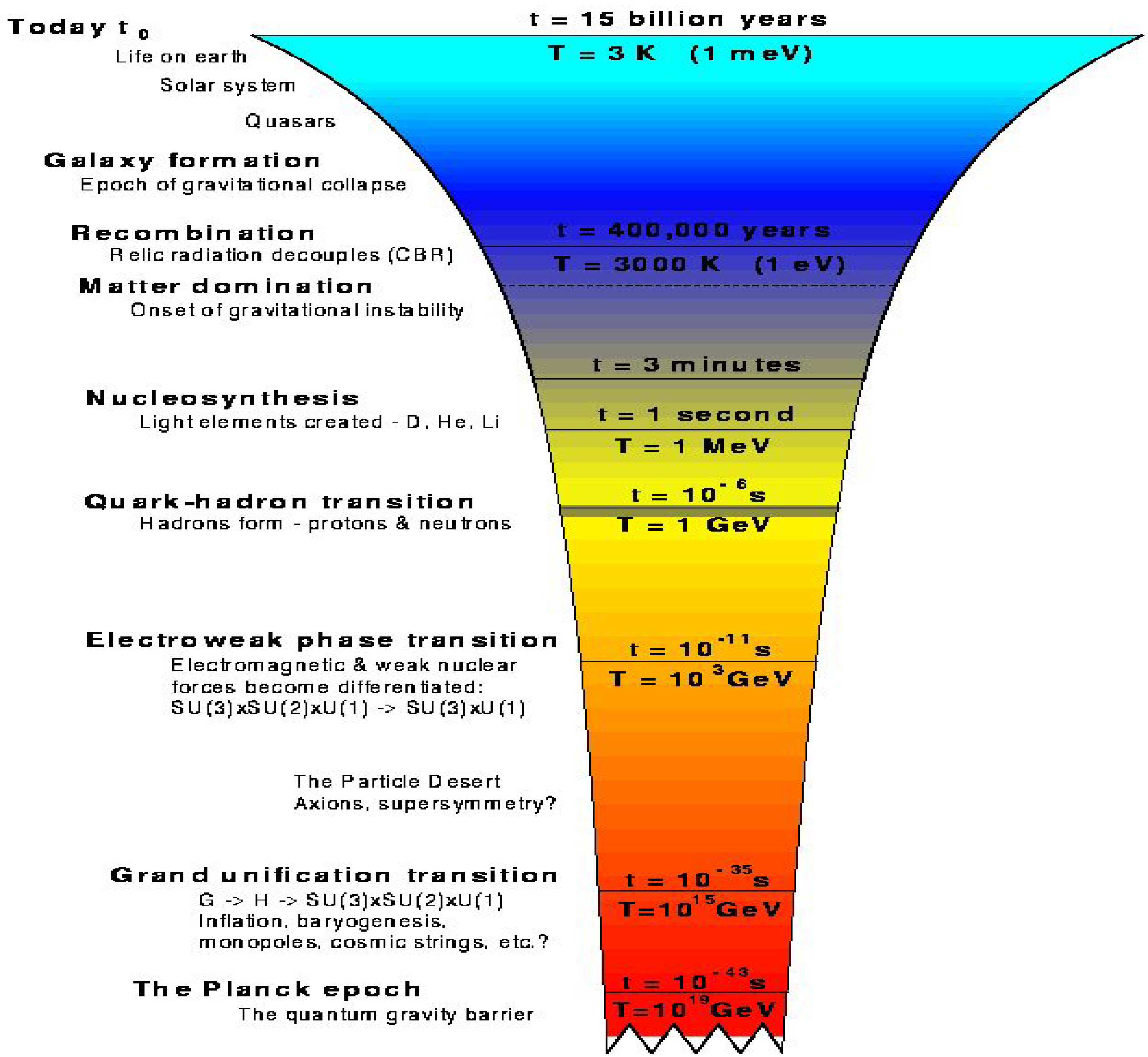
Paper in Preparation

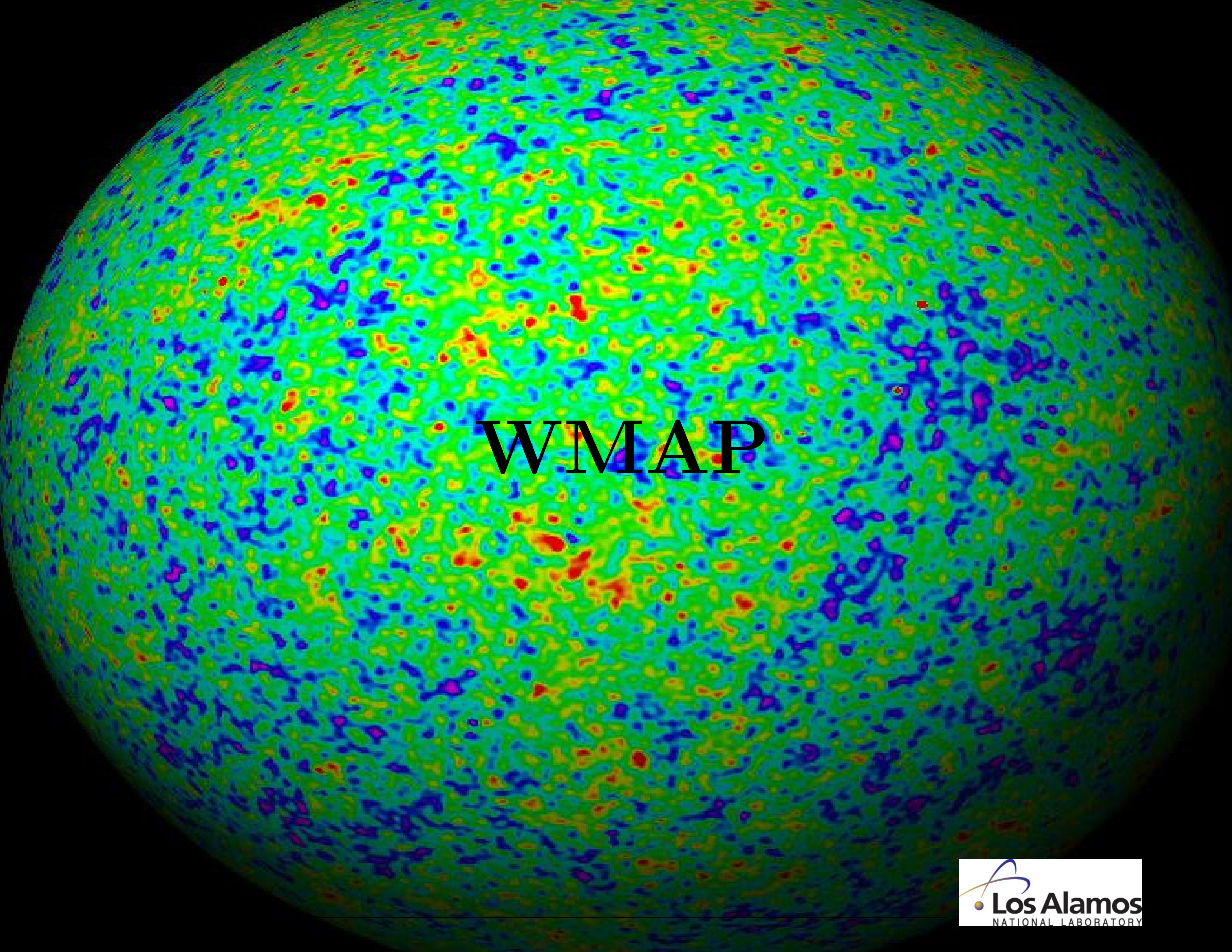

Today t_0
Life on earth
Solar system
Quasars
Galaxy formation
Epoch of gravitational collapse
Recombination
Relic radiation decouples (CBR)
Matter domination
Onset of gravitational instability
Nucleosynthesis
Light elements created - D, He, Li
Quark-hadron transition
Hadrons form - protons & neutrons
Electroweak phase transition
Electromagnetic & weak nuclear
forces become differentiated:
SU(3)xSU(2)xU(1) -> SU(3)xU(1)
The Particle Desert
Axions, supersymmetry?
Grand unification transition
G -> H -> SU(3)xSU(2)xU(1)
Inflation, baryogenesis,
monopoles, cosmic strings, etc.?
The Planck epoch
The quantum gravity barrier
t = 15 billion years
T = 3 K (1 meV)
t = 400,000 years
T = 3000 K (1 eV)
t = 3 minutes
t = 1 second
T = 1 MeV
$t = 10^{-6}$ s
T = 1 GeV
$t = 10^{-11}$ s
$T = 10^{3}$ GeV
$t = 10^{-35}$ s
$T=10^{15}$ GeV
$t = 10^{-43}$ s
$T=10^{19}$ GeV

The Very First Moment...

- Paradigm describing the very first moment: Inflation
- Predictions: spectrum of metric perturbations in scalar (density) and tensor (gravitational waves) sector
- Both cause anisotropies in CMB temperature (linear growth of fluctuations)
- Density fluctuations cause large-scale structure (LSS) formation (nonlinear growth of fluctuations)
- Aim: explain tiny ripples in CMBR across the sky and LSS we see in the Universe today
- Measure power-spectrum on large scales ($\to$ small k) from CMB
- Measure power-spectrum on small scales ($\to$ large k) from LSS

We can probe power spectrum of fluctuations
over about 4 decades in length scale
and almost entire age of the Universe!

- Take power spectrum today $\rightarrow$ process via transfer function $\rightarrow$ primordial spectrum
- Primordial spectrum $\rightarrow$ spectral index $\rightarrow$ k-dependence $\Rightarrow$ Information about inflation



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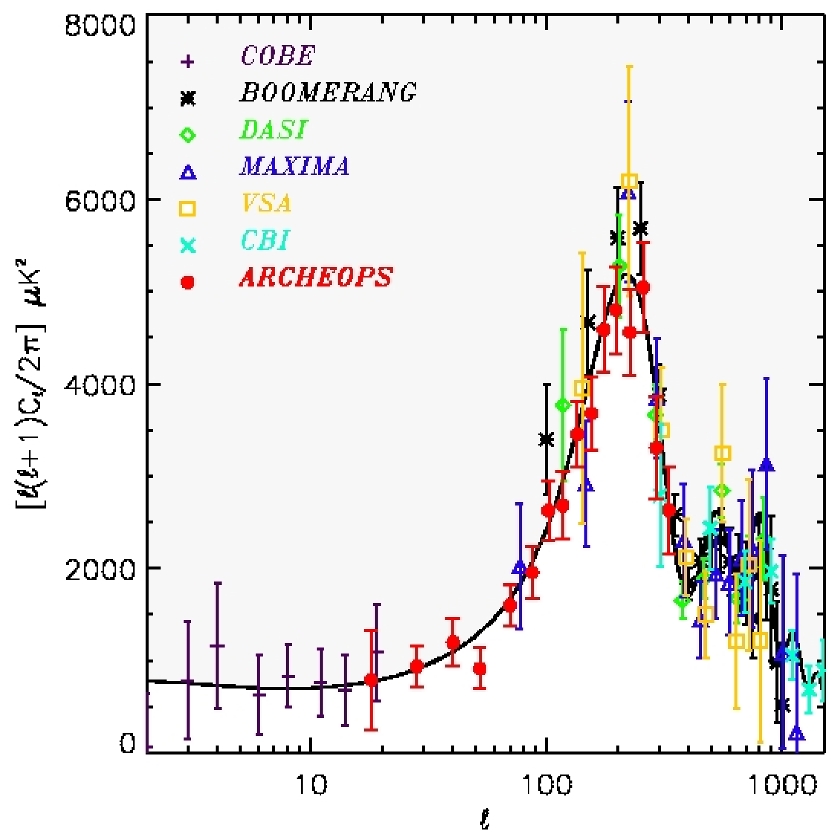
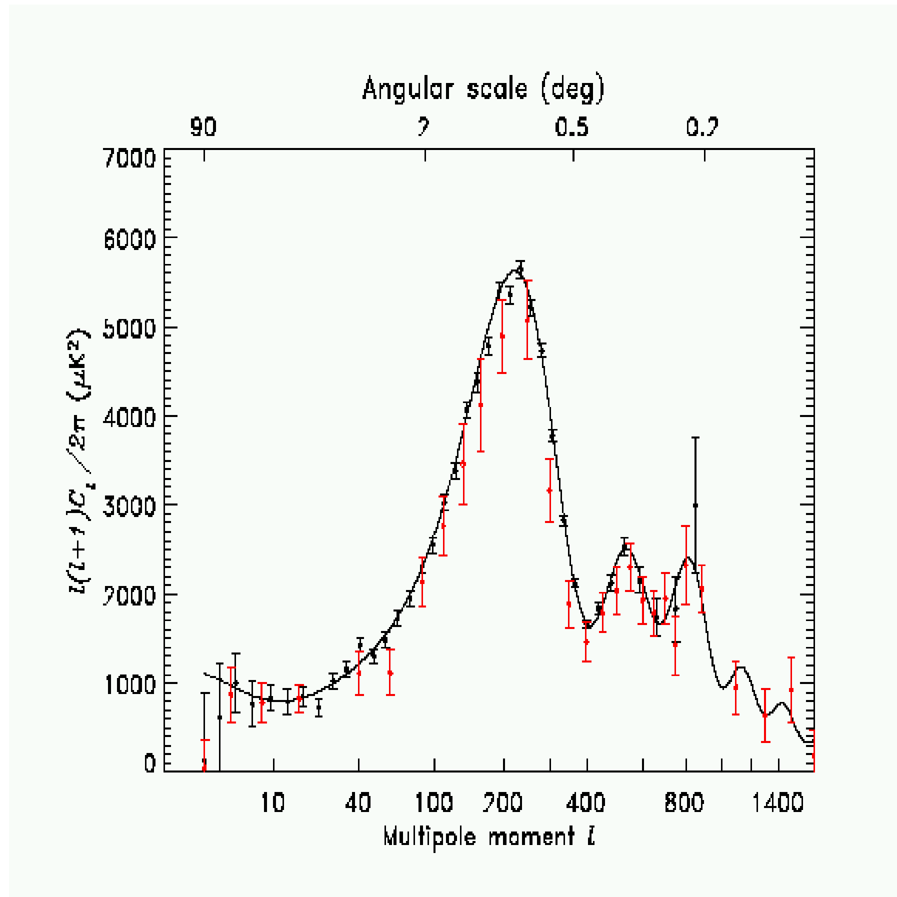
Angular Power Spectrum

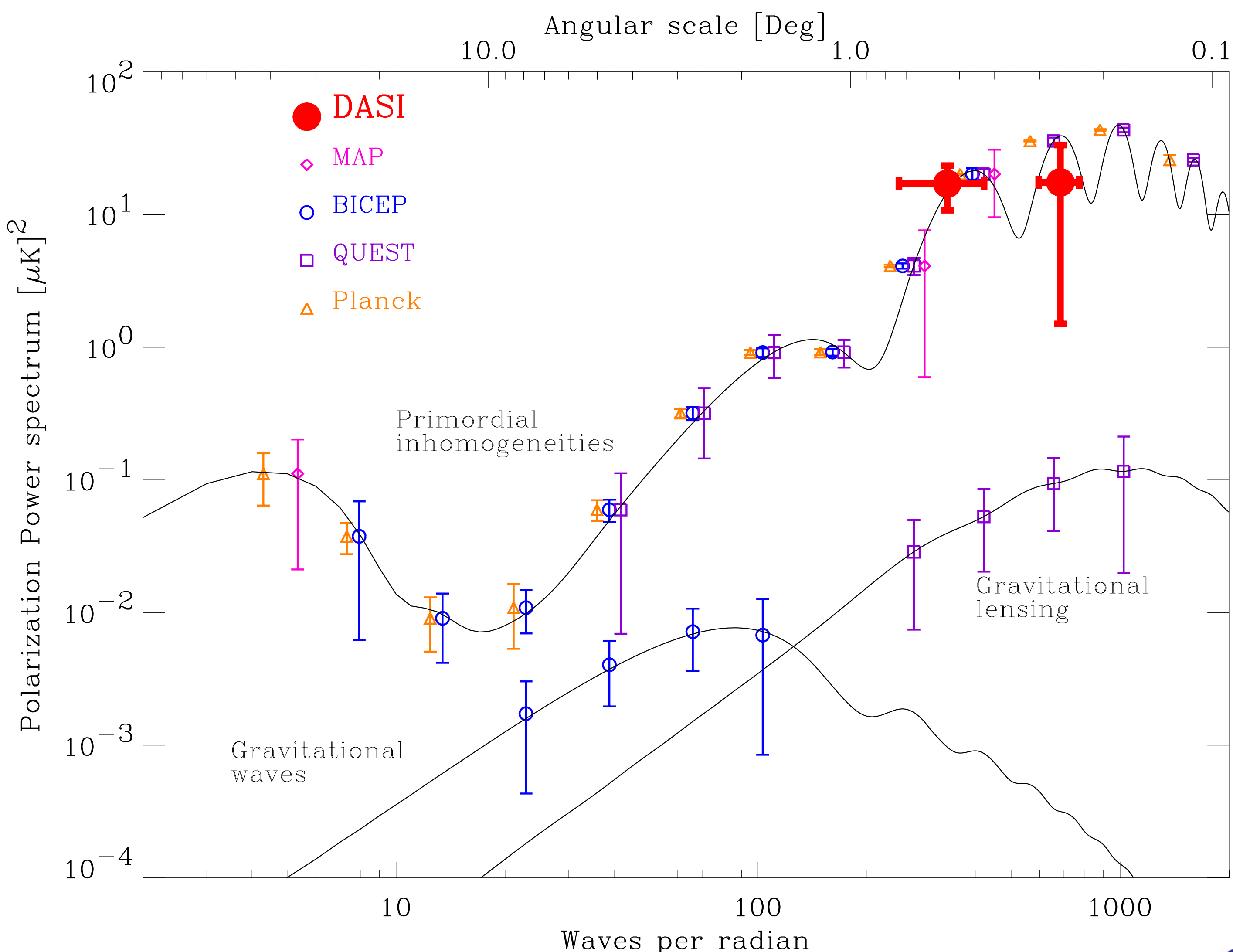


Before WMAP



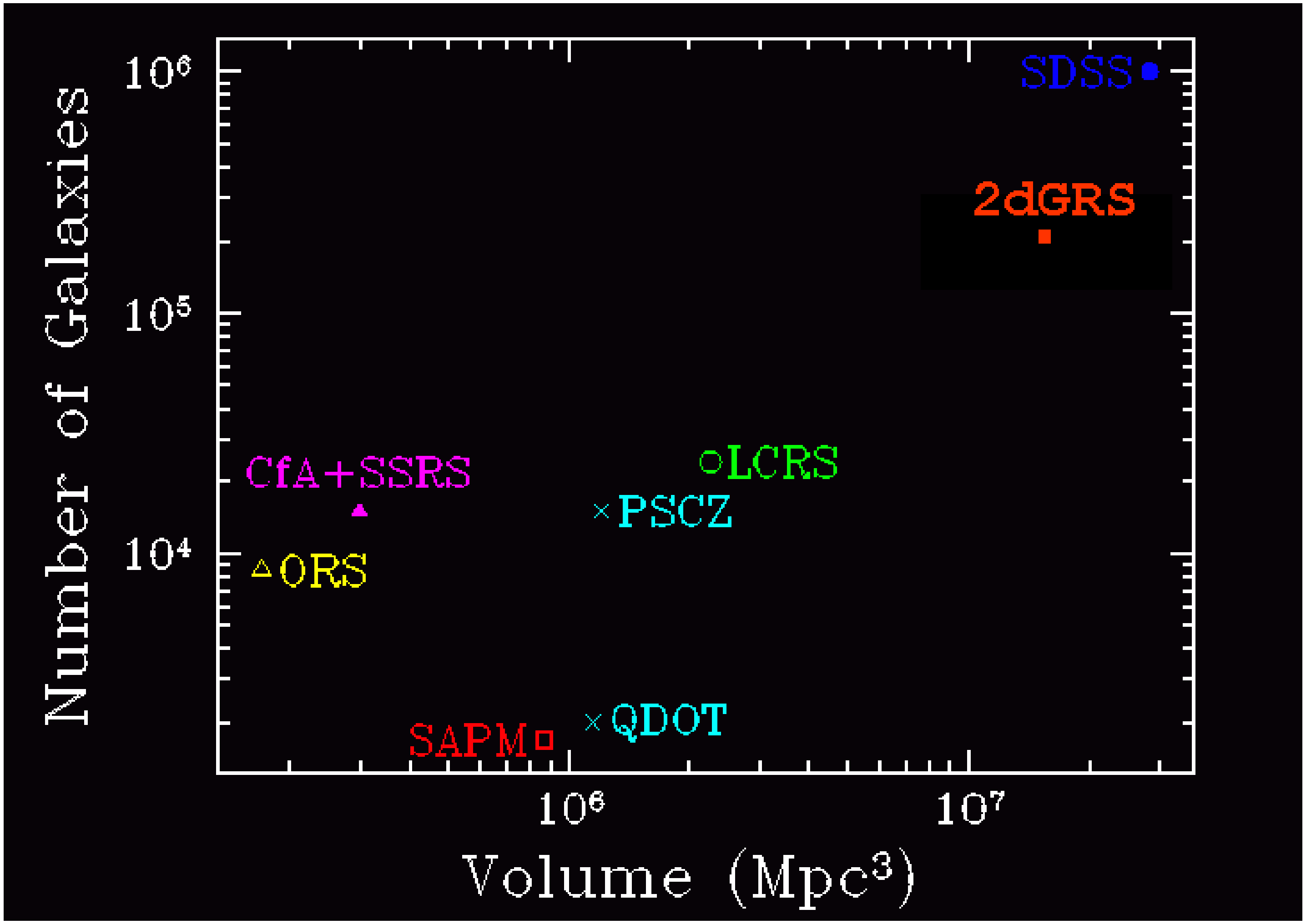
After WMAP

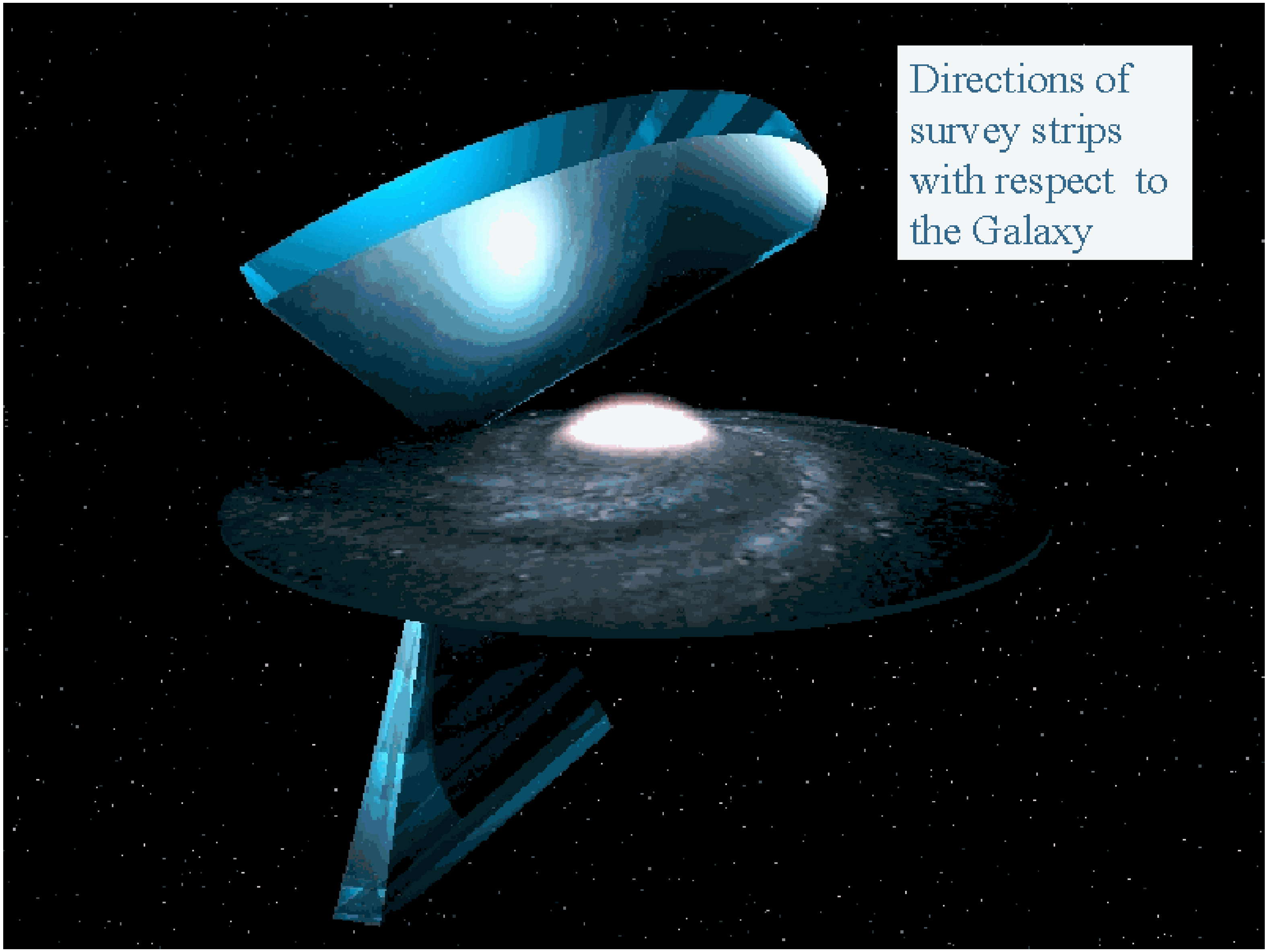
Polarization Power Spectrum - Presence and Future

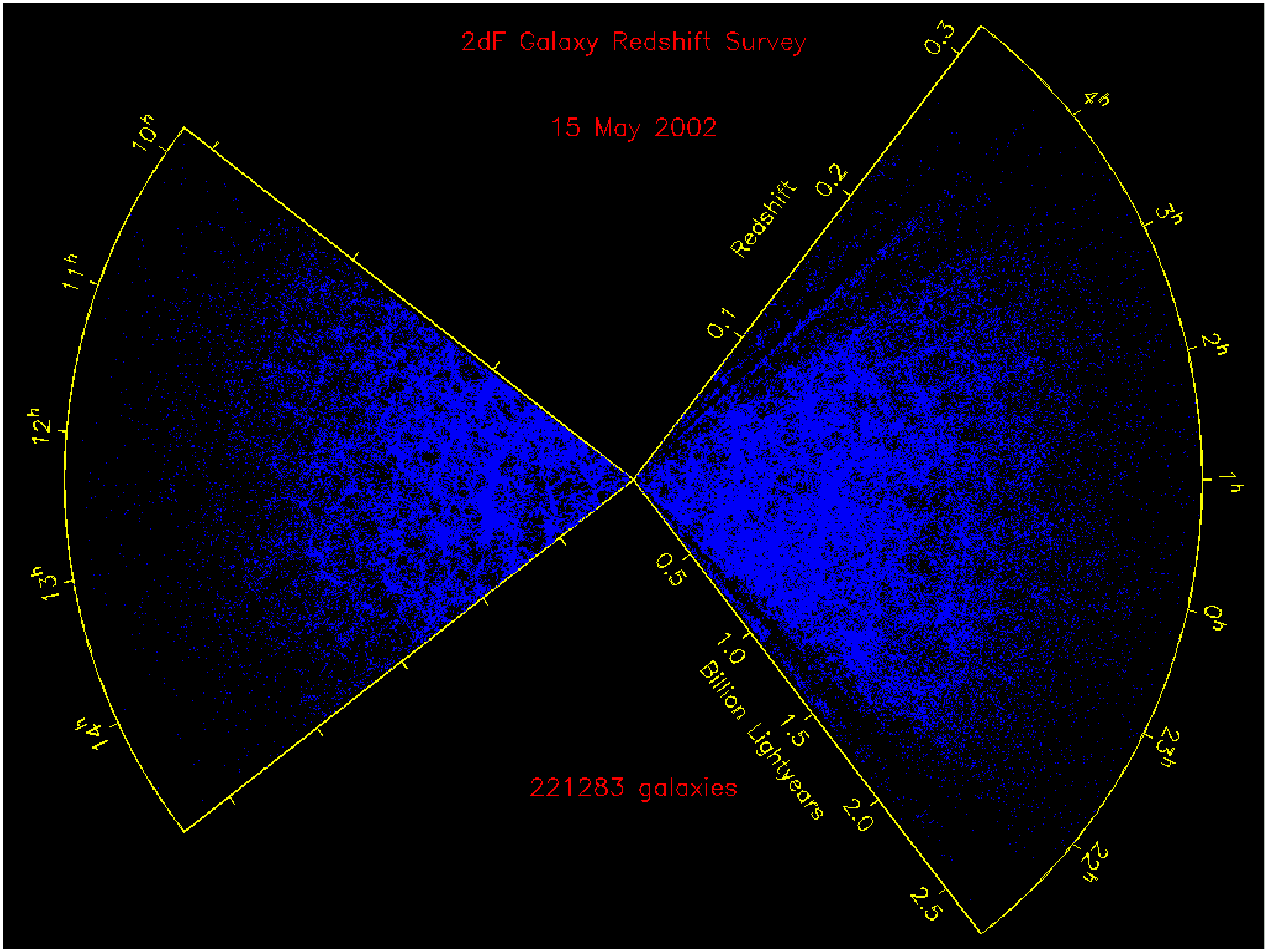


Large Scale Structure Surveys

Effective sizes of z-surveys

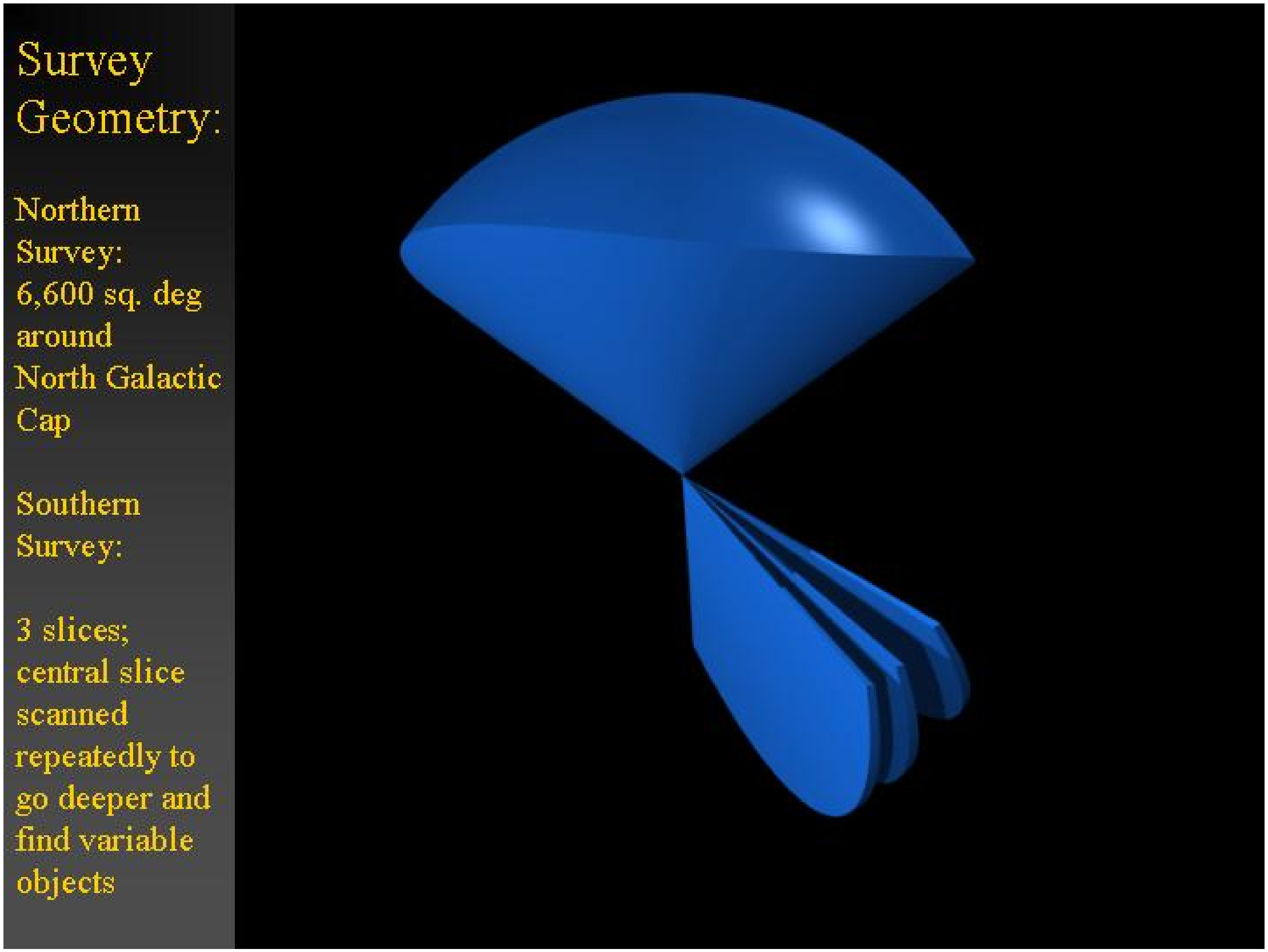



Directions of
survey strips
with respect to
the Galaxy



2dF Galaxy Redshift Survey
15 May 2002
221283 galaxies
Redshift
0.1
0.2
0.3
Billion Lightyears
0.5
1.0
1.5
2.0
2.5
10h
11h
12h
13h
14h
4h
3h
2h
1h
0h
23h
22h

The Sloan Digital Sky Survey



Some Facts on SDSS

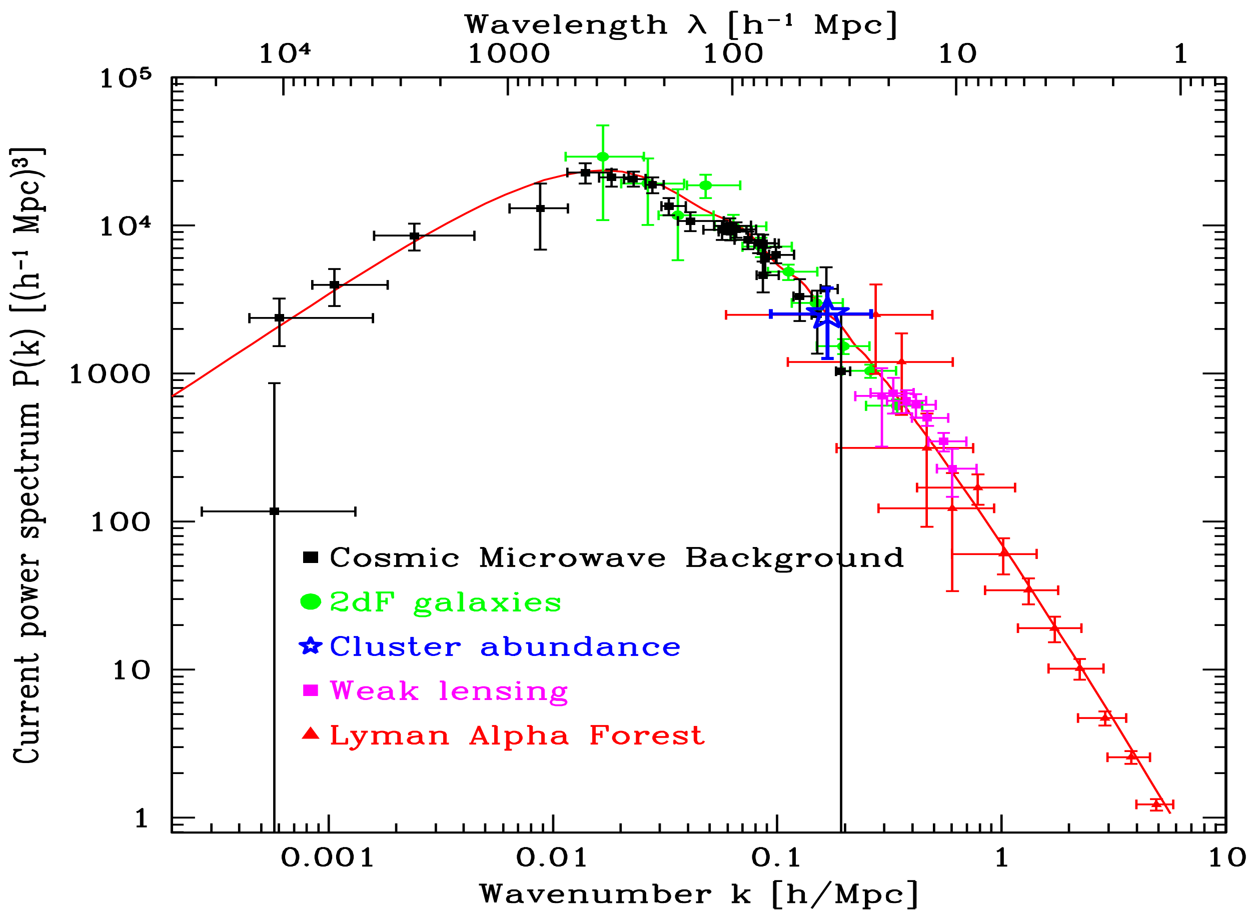
- Goal: 3-D map of the Universe over large volume

- Photometric Survey: $\sim 10^8$ 5-band CCD images
- Spectroscopic Survey: 10^6 galaxy and 10^5 QSO reshifts

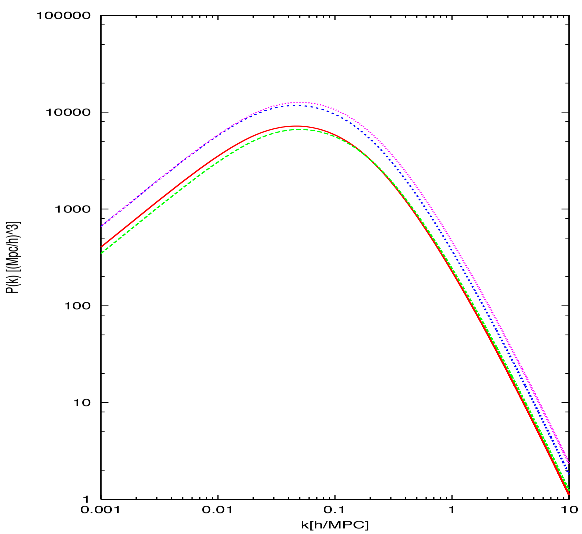
- Start: April 2000, End: June 2005

- Status: Imaging: 65% complete
  Spectroscopy: 43% complete

Linear Matter Power Spectrum

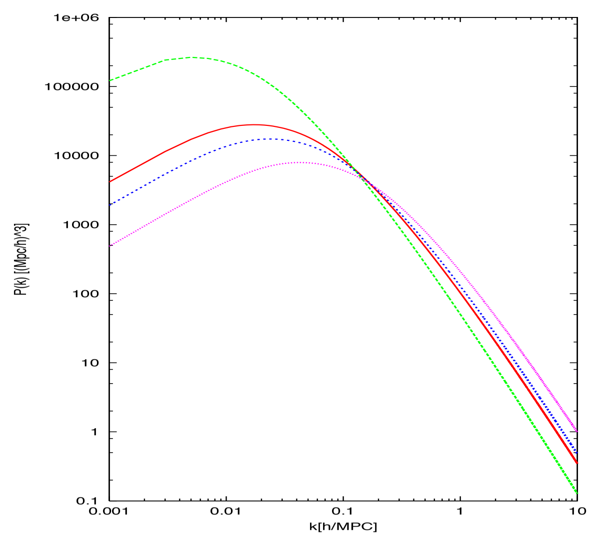


Linear Matter Power Spectrum for Different Cosmologies



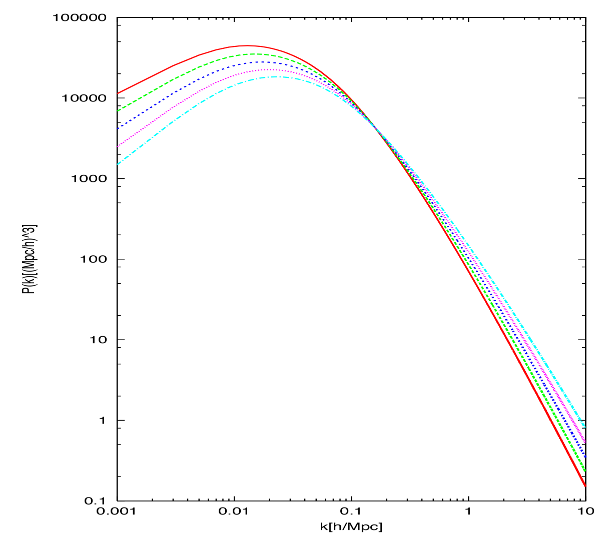
$h = 0.5$, $\Omega_m = 1.0$, $n_S = 1$ fixed;
red: $\Omega_B = 0.019$, σ_8-norm,
green: $\Omega_B = 0.0$, σ_8-norm,
blue: $\Omega_B = 0.019$, COBE,

purple: $\Omega_B = 0.0$, COBE



$h = 0.65$, $\Omega_B = 0.019$, $n_S = 1$ fixed;
red: $\Omega_m = 0.3 \Rightarrow \sigma_8 = 1.0$,
green: $\Omega_m = 0.1 \Rightarrow \sigma_8 = 0.4$,
blue: $\Omega_m = 0.4 \Rightarrow \sigma_8 = 1.2$,

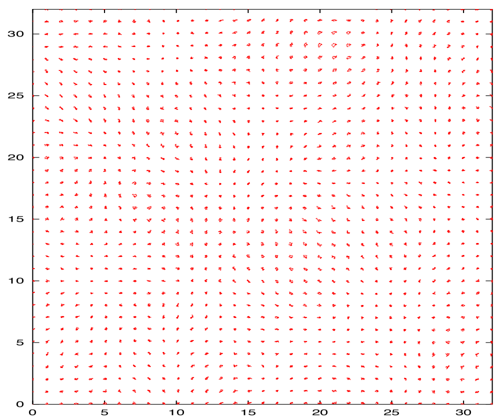
purple:$\Omega_m = 0.7 \Rightarrow \sigma_8 = 1.5$



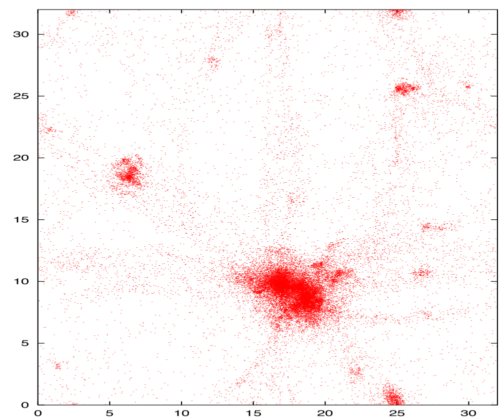
$h = 0.65$, $\Omega_m = 0.3$, $\Omega_B = 0.019$,
fixed;
red: $n_S = 0.8 \Rightarrow \sigma_8 = 1.2$,
green: $n_S = 0.9 \Rightarrow \sigma_8 = 1.1$,
blue: $n_S = 1.0 \Rightarrow \sigma_8 = 1.0$,
purple:$n_S = 1.1 \Rightarrow \sigma_8 = 0.9$,

light-b.:$n_S = 1.2 \Rightarrow \sigma_8 = 0.8$

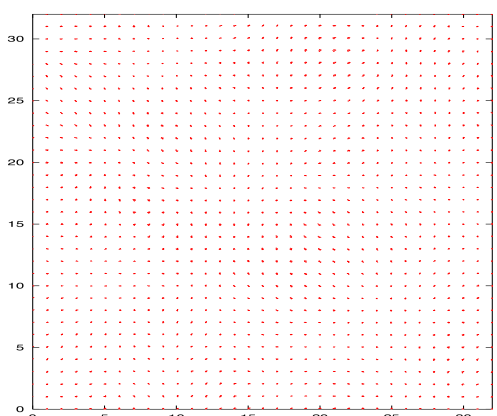
First Results from MC^2



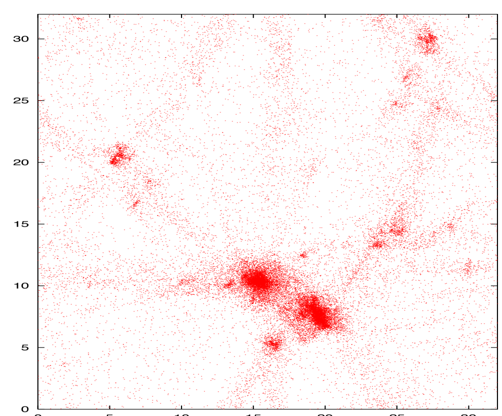
SCDM, $z = 100$



SCDM, $z = 0$



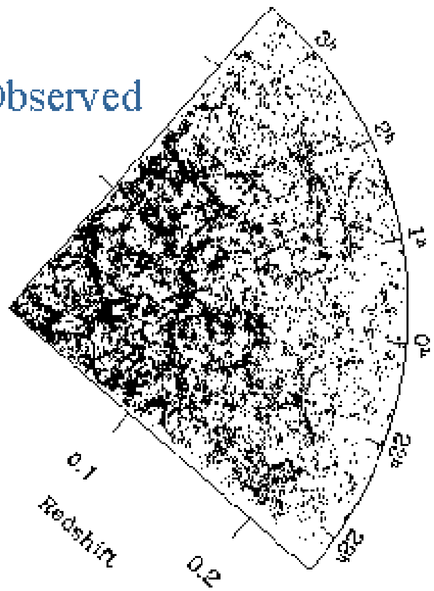
ΛCDM, $z = 100$



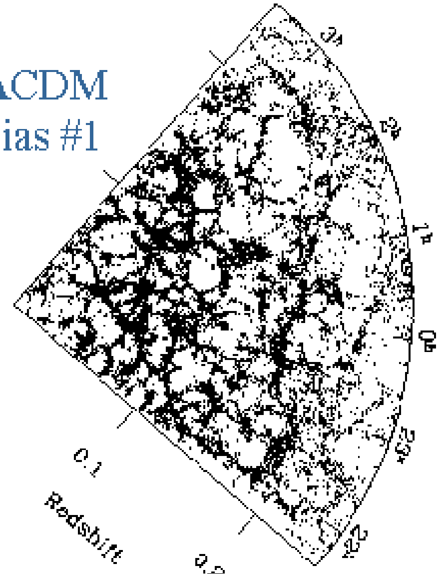
ΛCDM, $z = 0$

Models vs observations

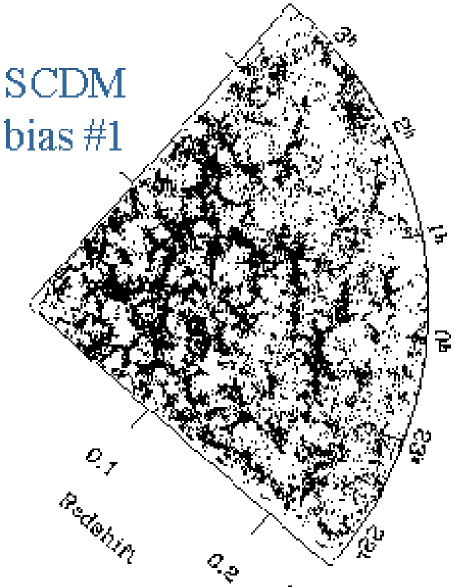
Observed



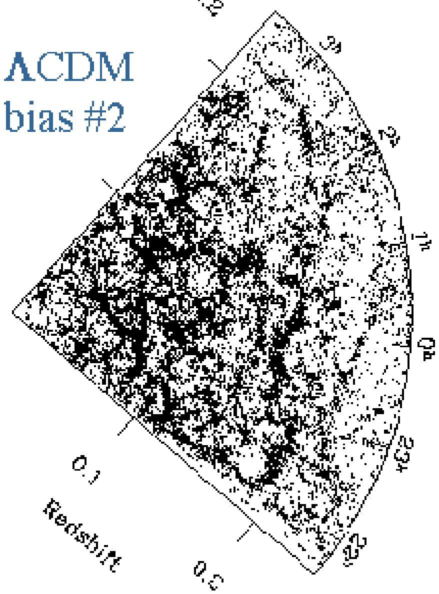
ΛCDM bias #1



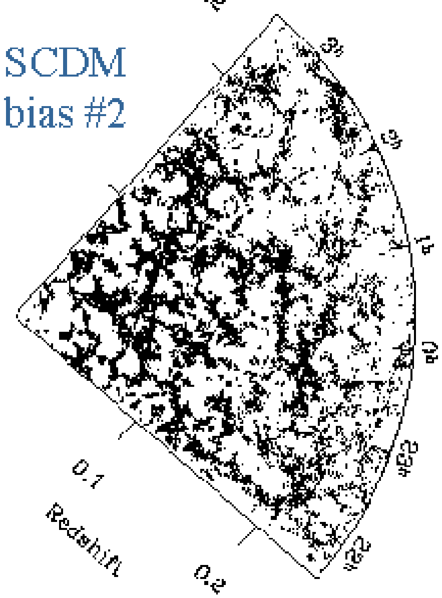
SCDM bias #1



ΛCDM bias #2



SCDM bias #2

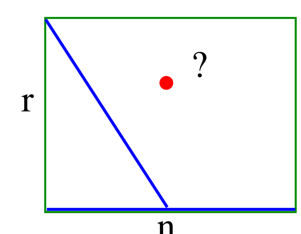


More Concrete . . .

- Power Spectrum: $\langle 0|\hat{u}(\eta,\mathbf{x})\hat{u}(\eta,\mathbf{x}+\mathbf{r})|0\rangle = \int_0^\infty \frac{dk}{k}\frac{\sin kr}{kr}P_u(\eta,k)$
  - or $P_S(k) = \frac{k^3}{2\pi^2}|\frac{u_k(\eta)}{z}|^2$, $z = \frac{a\dot{\phi}}{H}$ $\quad P_T(k) = \frac{k^3}{2\pi^2}|\frac{v_k(\eta)}{a}|^2$
    - Spectral Indices: $n_S = 1 + \frac{d\ln P_S(k)}{d\ln k}$ $\quad n_T = \frac{d\ln P_T(k)}{d\ln k}$
- Ratio of the quadrupole moments: $r = \frac{C_2^T}{C_2^S}$, running of n_S: $\frac{dn_S}{d\ln k}$

- COBE: $n_S = 1.2 \pm 0.3$ BOOMERANG: $n_S = 0.93^{+0.1}_{-0.08}$
- PLANCK: $\Delta n_S \approx 1\%$, $\Delta r \approx 0.05$ CMBPOL: $\Delta n_S \approx 1\%$, $\Delta r \approx 0.001$
- Typical Inflation $\Delta n_S \approx 5\%$ $(10 \le l \le 1500)$, $\Delta r \le 0.001$

- Simplified: $P_k \propto k^n \Rightarrow n_T = n_S - 1,\ r = 7(1 - n_S)$



The Background Equations

$$H^2 = \frac{1}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$
$$\dot{H} = -\frac{1}{2}\dot{\phi}^2$$
$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

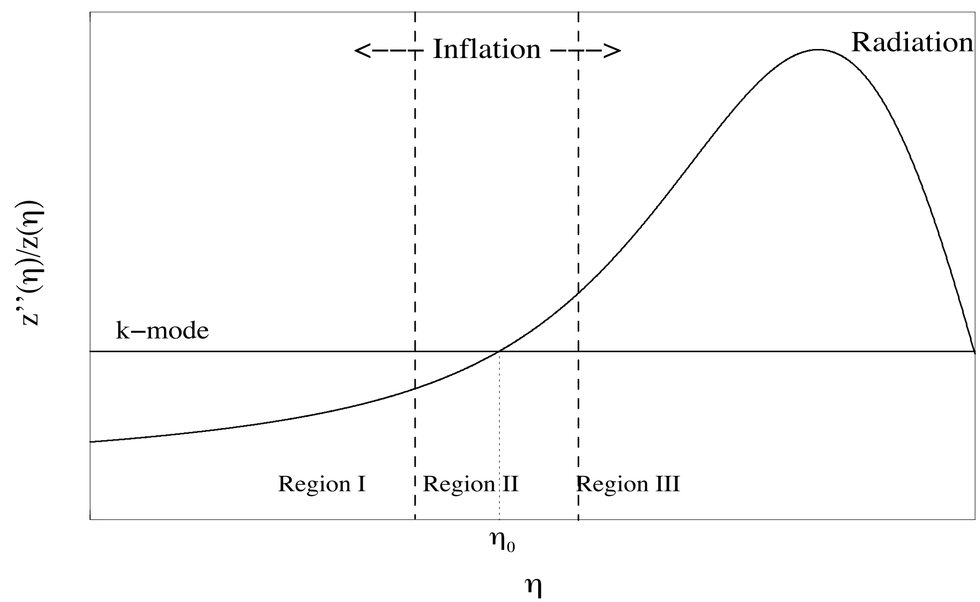
Dynamical Equations for Perturbations

$$u_k''(\eta) + \left[k^2 - \frac{z''}{z}\right] u_k(\eta) = 0$$
$$v_k''(\eta) + \left[k^2 - \frac{a''}{a}\right] v_k(\eta) = 0$$

with

$$z''/z = 2a^2H^2\left(1 + \frac{\dot{H}}{2H^2} + \frac{\dot{H}V_\phi}{H^3\dot{\phi}} - \frac{\ddot{H}}{2H^3} + \frac{\dot{H}^2}{H^4} - \frac{V_{\phi\phi}}{2H^2}\right)$$
$$a''/a = 2a^2H^2\left(1 - \frac{\dot{H}}{2H^2}\right)$$

Perturbations

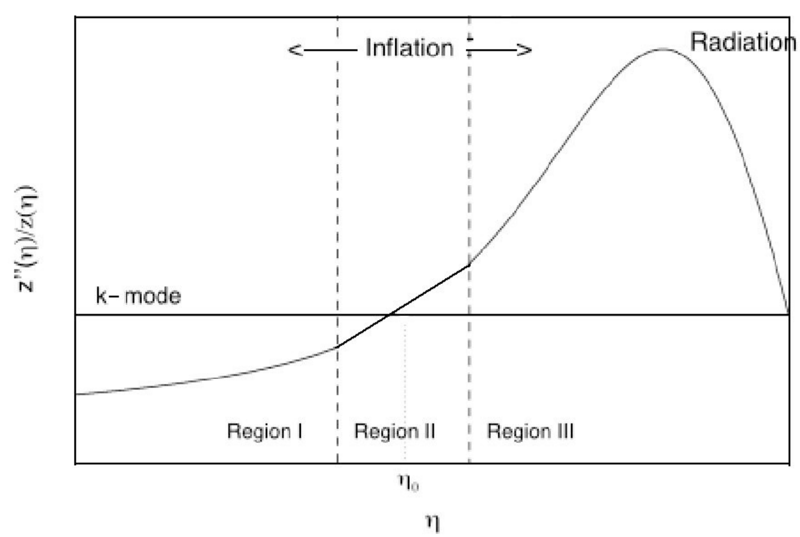


The Simplest Approximation

- Take limit $k \to \infty \Rightarrow u_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}$
- Take limit $k \to 0 \Rightarrow u_k(\eta) = C_k z$
- Match both solutions at $k\eta = 1$, arbitrary point!

The Slow-Roll Improvement

- Improve intermediate region by "best-fit-powerlaw" $\Rightarrow \frac{z''}{z} = \frac{1}{\eta^2}\left(c - \frac{1}{4}\right)$
  - Match intermediate solution to exact solution

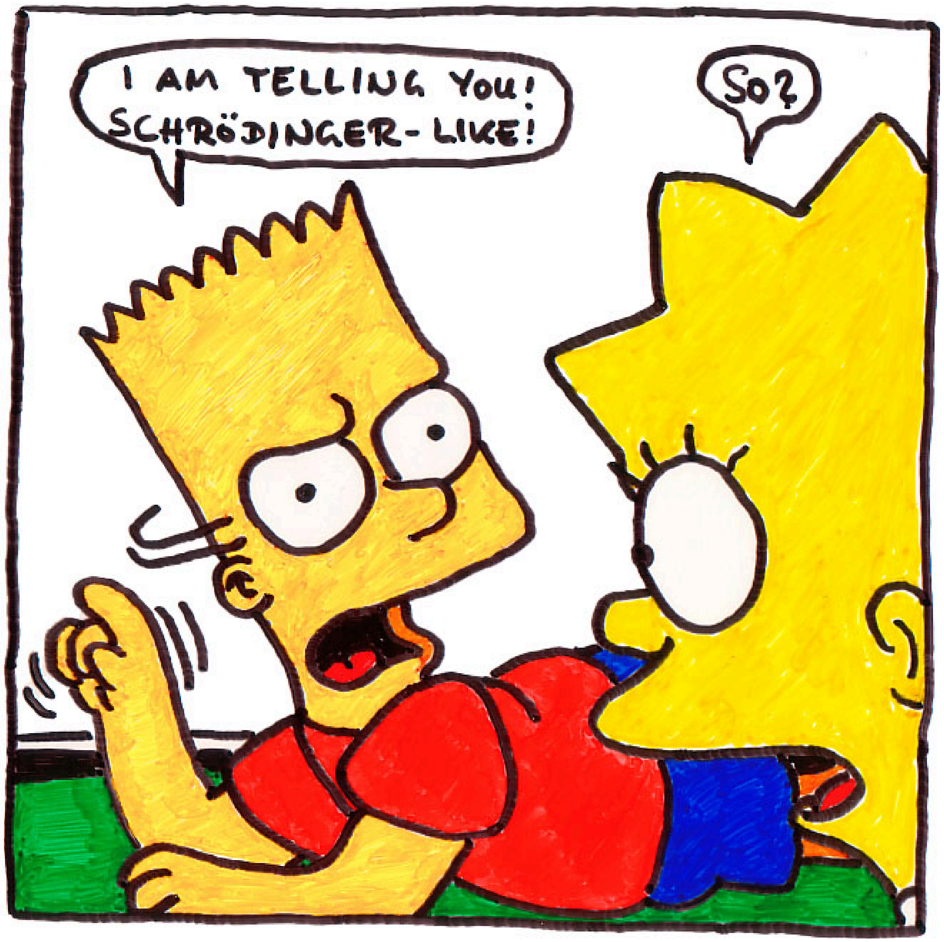


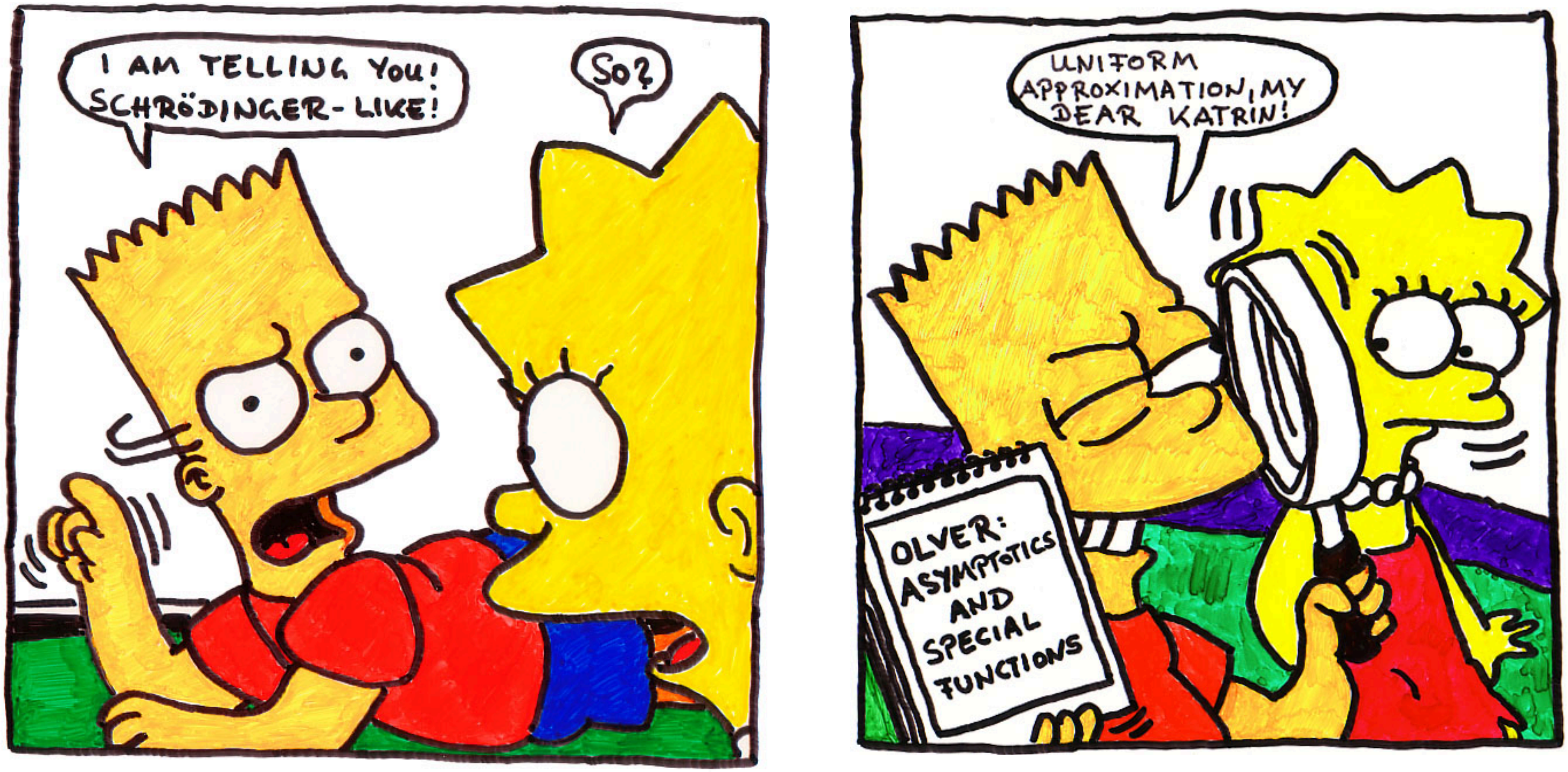
- Control of approximation?
- Systematic improvement beyond leading order?
- Matching problem

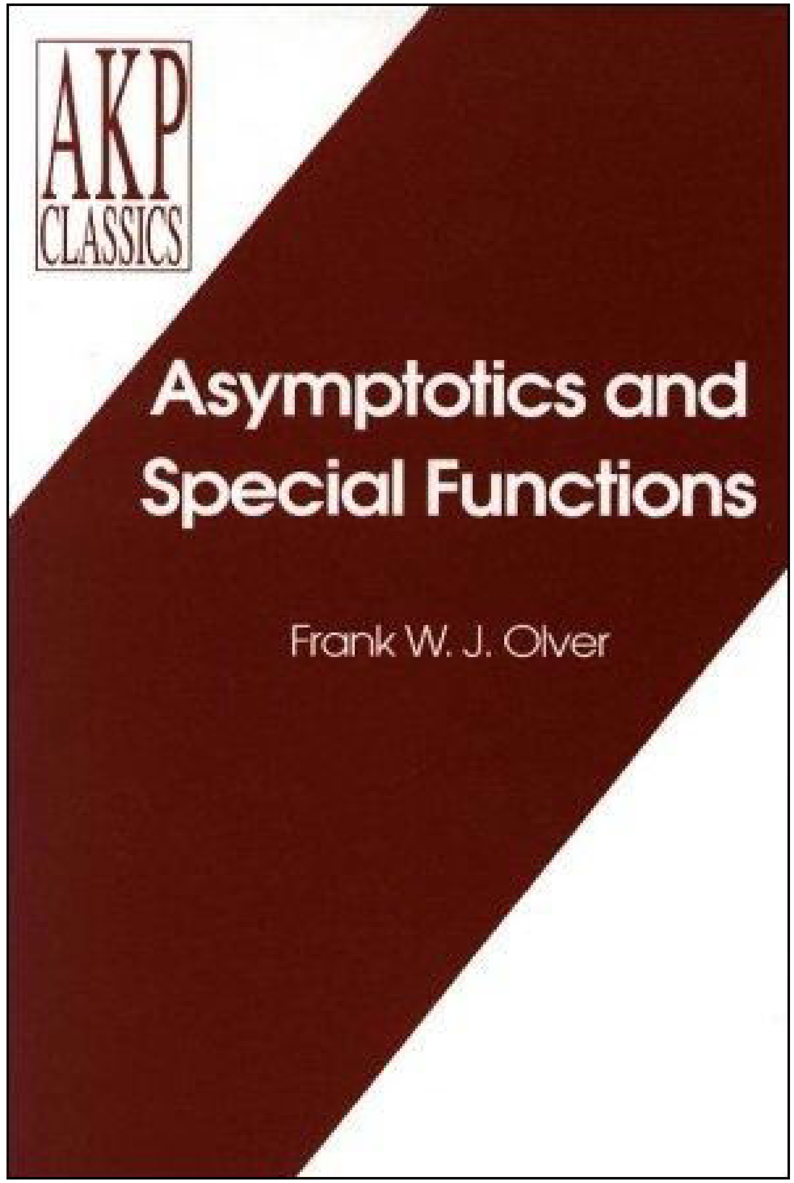
The Perturbation Equations

$$u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k = 0$$
$$\Rightarrow u_k''(\eta) + \left[k^2 - \frac{1}{\eta^2}\left(\nu_S^2(\eta) - \frac{1}{4}\right)\right] u_k(\eta) = 0$$

- Find approximation which allows time-dependent ν
- Find systematically improvable approximation
- Use approximation with well-defined error bounds
- Find approximation valid over whole range of interest
- Normalize approximation by matching to $k \to \infty$ limit


I AM TELLING YOU! SCHRÖDINGER-LIKE!
SO?


I AM TELLING YOU! SCHRÖDINGER-LIKE!
SO?
UNIFORM APPROXIMATION, MY DEAR KATRIN!
OLVER: ASYMPTOTICS AND SPECIAL FUNCTIONS


AKP
CLASSICS
Asymptotics and
Special Functions
Frank W. J. Olver

The Uniform Approximation

- To solve: $d^2w/dx^2 - [u^2 f(x) + g(x)]w = 0$
- Form of approximation depends on # and nature of transition points (= zeros or turning points x_0 and singularities) of $f(x)$
- Singularity leads to WKB-like solution
- In case of interest $u^2 f(x) + g(x) \rightarrow k^2 - [\nu^2(\eta) - 1/4]/\eta^2$

  $\Rightarrow$ turning point and singularity
- Introduce new variables ζ and W: $\zeta\left(\frac{d\zeta}{dx}\right)^2 = f(x)$, $w = \left(\frac{d\zeta}{dx}\right)^{-1/2} W$

$\Rightarrow \frac{2}{3}\zeta^{3/2} = \int_{x_0}^{x} f^{1/2}(t)dt$ for $x \geq x_0$, $\frac{2}{3}(-\zeta)^{3/2} = \int_{x_0}^{x}[-f(t)]^{1/2}dt$ for $x \leq x_0$

$$\Rightarrow d^2W/d\zeta^2 = [u^2\zeta + \psi(\zeta)]W$$

with $\psi(\zeta) = \frac{5}{16\zeta^2} + \left[4f(x)f''(x) - 5f'^2(x)\right]\frac{\zeta}{16f^3(x)} + \frac{\zeta g(x)}{f(x)}$

- Essence of aproximate procedure: in first order disregard $\psi(\zeta)$

  $\Rightarrow$ Solution can be written in Ai- and Bi-functions

- Solution: $w_1(u,x) = \left(\frac{f(x)}{\zeta}\right)^{-1/4}\left[\mathrm{Bi}(u^{2/3}\zeta) + \epsilon_1(u,x)\right],$

$$w_2(u,x) = \left(\frac{f(x)}{\zeta}\right)^{-1/4}\left[\mathrm{Ai}(u^{2/3}\zeta) + \epsilon_2(u,x)\right]$$

where $|\epsilon_1(u,x)| \leq E(u^{2/3}\zeta)M(u^{2/3}\zeta)/\lambda\left[\exp\left\{\frac{\lambda\mathcal{V}_{a,x}(H)}{u}\right\} - 1\right]$

$$|\epsilon_2(u,x)| \leq E^{-1}(u^{2/3}\zeta)M(u^{2/3}\zeta)/\lambda\left[\exp\left\{\frac{\lambda\mathcal{V}_{x,b}(H)}{u}\right\} - 1\right]$$

Error-Control-Function: $H(x) = \int_{x_0}^{x}\left[\frac{1}{|f|^{1/4}}\frac{d^2}{dx^2}\left(\frac{1}{|f|^{1/4}}\right) - \frac{g}{|f|^{1/2}} - \frac{5|f|^{1/2}}{16|\zeta|^3}\right]dx$

E, M, λ auxiliary functions of Ai- and Bi-functions

Case of pure singularity

- Solution: $w_1(x) = f^{-1/4}(x)\exp\left\{\int f^{1/2}(x)dx\right\}\{1+\epsilon_1(x)\}$

$$w_2(x) = f^{-1/4}(x)\exp\left\{-\int f^{1/2}(x)dx\right\}\{1+\epsilon_2(x)\}$$

with $|\epsilon_j(x)| \leq \exp\left\{\frac{1}{2}\mathcal{V}_{aj,x}(F)\right\} - 1 \quad (j=1,2)$

$$F(x) = \int\left[\frac{1}{f^{1/4}}\frac{d^2}{dx^2}\frac{1}{f^{1/4}} - \frac{g}{f^{1/2}}\right]dx$$

IMPORTANT: In order for $\mathcal{V}(F)$ to converge

at an endpoint singularity we have to choose $g(x) = -\frac{1}{4x^2}$

Outline

- Pick $f(x) \to f_S(\eta) = \frac{\nu^2(\eta)}{\eta^2} - k^2$, $g(x) \to g_S(\eta) = -\frac{1}{4\eta^2}$, $u = 1$
- Calculate the unnormalized solution for u_k on the left and right of the turning point
- Take the limit $-k\eta \to \infty$ and normalize u_k to desired vacuum state

  Notice: Usually one picks Bunch-Davies vacuum $u_k \to \frac{e^{-ik\eta}}{\sqrt{2k}}$

  but one could pick also other vacuum states

  $\Rightarrow$ Trans-Planckian Effects
- Take limit $k\eta \to 0^-$, regime of interest for power spectrum

  Recover WKB-form of solution
- Calculate power spectra and spectral indices
- Calculate error bounds

- From $f_S(\eta) = \frac{\nu^2(\eta)}{\eta^2} - k^2$ find turning point at $k^2 = \nu_S^2/\eta^2$
  - Find solution on the left and the right of turning point

- Unnormalized solution $u_{k\underset{>}{<}}^{(1,2)}(\eta) = \zeta_{\underset{>}{<}}^{1/4}(\eta)\,[f_S(\eta)]^{-1/4}\,\mathrm{Ai}^{(1,2)}[\zeta_{\underset{>}{<}}(\eta)]$

  with $\zeta_{\underset{>}{<}}(\eta) = \mp\left\{\pm\frac{3}{2}\int_\eta^{\bar{\eta}_S} d\eta'\,[\mp f_S(\eta)]^{1/2}\right\}^{2/3}$

  - General solution $u_k = Au_k^{(1)} + Bu_k^{(2)}$

- Fix A and B ensuring that $u_k(\eta) = (1/\sqrt{2k})e^{-ik\eta}$ in the limit $k \to \infty$

  $\Rightarrow A = \sqrt{\pi/2},\ B = -i\sqrt{\pi/2}$

  - For power spectrum: $k\eta \to 0^-$ limit, simplifies $u_k(\eta)$ a lot
    - $u_k(\eta) \overset{k\eta\to 0^-}{=} -i\sqrt{-\frac{\eta}{\nu_S(\eta)}}\exp\left(\int_{\bar{\eta}_S}^{\eta} d\eta'\sqrt{f_S(\eta)}\right)$

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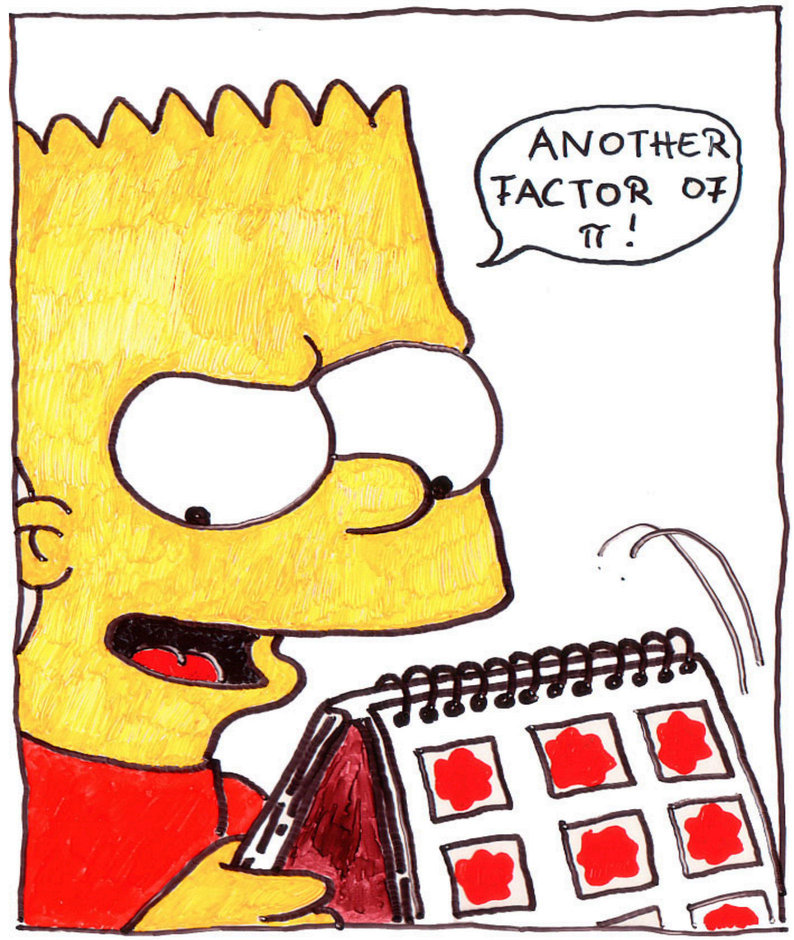

ANOTHER FACTOR OF π!

Results

- Power-Spectrum:

$$P_S(k) \equiv (k^3/2\pi^2)\left|\frac{u_k(\eta)}{z(\eta)}\right|^2 = \frac{k^3}{4\pi^2}\frac{1}{|z(\eta)|^2}\frac{-\eta}{\nu_S(\eta)}\exp\left(2\int_{\bar{\eta}_S}^{\eta} d\eta' \sqrt{f_S(\eta')}\right)$$

- Spectral indices:

$$n_S(k) = 1 + \frac{d\ln P_S(k)}{d\ln k} = 4 + 2\frac{\nu_S(\bar{\eta}_S)k}{|\bar{\eta}_S|}\int_{\bar{\eta}_S}^{\eta}\frac{d\eta'}{\sqrt{f_S(\eta')}},$$

$$n_T(k) = \frac{d\ln P_T(k)}{d\ln k} = 3 + 2\frac{\nu_T(\bar{\eta}_T)k}{|\bar{\eta}_T|}\int_{\bar{\eta}_T}^{\eta}\frac{d\eta'}{\sqrt{f_T(\eta')}}$$

$\bar{\eta}$: calculated at turning point

- To obtain local expression for n_S, n_T: further approximation

- Integrand has square-root singularity at $\bar{\eta}$, vanishes linearly at upper limit η

$\Rightarrow$ Dominant contribution from $\bar{\eta} \Rightarrow$ Taylor expansion of $\nu^2(\eta)$ around $\bar{\eta}$

$$n_S(k) = 4 - 2\nu_S(\bar{\eta}_S)\left[1 - \left(1 - \frac{\pi}{2}\right)\frac{\nu_S'(\bar{\eta}_S)}{\nu_S(\bar{\eta}_S)}\bar{\eta}_S\right]$$

$$n_T(k) = 3 - 2\nu_T(\bar{\eta}_T)\left[1 - \left(1 - \frac{\pi}{2}\right)\frac{\nu_T'(\bar{\eta}_T)}{\nu_T(\bar{\eta}_T)}\bar{\eta}_T\right]$$

Comparison with Slow-Roll

- Introduction of slow-roll parameters:

$$\epsilon \equiv -\frac{\dot{H}}{H^2},\ \delta \equiv -\frac{\ddot{\phi}}{(H\dot{\phi})},\ \text{and } \xi_2 \equiv \frac{(\dot{\epsilon}-\dot{\delta})}{H}$$

$$\Rightarrow \frac{z''}{z} = 2a^2H^2\left(1+\epsilon-\frac{3}{2}\delta-\frac{1}{2}\epsilon\delta+\frac{1}{2}\delta^2+\frac{1}{2}\xi_2\right), \quad \frac{a''}{a} = 2a^2H^2\left(1-\frac{1}{2}\epsilon\right)$$

- Expand ν_S, ν_T, $\eta\nu'_S$, $\eta\nu'_T$ in slow-roll parameters

$$\Rightarrow n_S(k) \simeq 1-4\epsilon_0+2\delta_0+2\left(\frac{17}{3}-\pi\right)\epsilon_0\delta_0-2\left(\frac{20}{3}-\pi\right)\epsilon_0^2-2\left(\frac{4}{3}-\frac{\pi}{2}\right)\xi_{02}$$

$$n_T(k) \simeq -2\epsilon_0-2\left(\frac{23}{4}-\pi\right)\epsilon_0^2+2\left(\frac{14}{3}-\pi\right)\epsilon_0\delta_0$$

- Standard slow-roll expressions:

$$\Rightarrow n_S(k) \simeq 1-4\epsilon+2\delta+(6+10C)\epsilon\delta-8(C+1)\epsilon^2-2C\xi_2$$

$$n_T(k) \simeq -2\epsilon-2(3+2C)\epsilon^2+4(1+C)\epsilon\delta \quad C=-2+\ln 2+\gamma \simeq -0.73$$

- Notice: slow-parameters are calculated at different points ($\rightarrow$ E. Stewart)
- Our results correspond to a resummation of the slow-roll expansion to all orders!

Example: ν=Constant

- ν=const. includes different inflationary models,

  e.g. power-law inflation, inflation near the maximum

  - Advantage: exact results are available

- Power-Spectrum: $P_S^{\rm ex} = \frac{1}{\pi^3} 2^{2\nu_S - 3}\Gamma^2(\nu_S) k^2 (-k\eta)^{-2\nu_S+1}\left(\frac{H}{a\dot\phi}\right)^2$

$$n_S = 4 - 2\nu_S, \quad n_T = 3 - 2\nu_T$$

  - Uniform approximation:

$$P_S^{\rm ex}(k) = \frac{2^{2\nu-2}}{\pi^2} e^{-2\nu}\nu^{2\nu-1}\left(\frac{H}{a\dot\phi}\right)^2 (-k\eta)^{1-2\nu}k^2\left(1 + \frac{1}{6\nu_S} + \frac{1}{72\nu_S^2} + \cdots\right)$$

$$P_S^{1}(k) = \frac{2^{2\nu-2}}{\pi^2} e^{-2\nu}\nu^{2\nu-1}\left(\frac{H}{a\dot\phi}\right)^2 (-k\eta)^{1-2\nu}k^2$$

$$P_S^{2}(k) = \frac{2^{2\nu-2}}{\pi^2} e^{-2\nu}\nu^{2\nu-1}\left(\frac{H}{a\dot\phi}\right)^2 (-k\eta)^{1-2\nu}k^2\left(1 + \frac{1}{6\nu_s}\right)$$

Spectral indices exact already in leading order!

  - Error bounds: $|\epsilon_{1,2}| \leq \sqrt{2}\left(\frac{1}{6\nu_S} + \frac{1.04}{72\nu_S^2} + \cdots\right)$

- Slow-roll approximation gives only expanded version of spectral indices

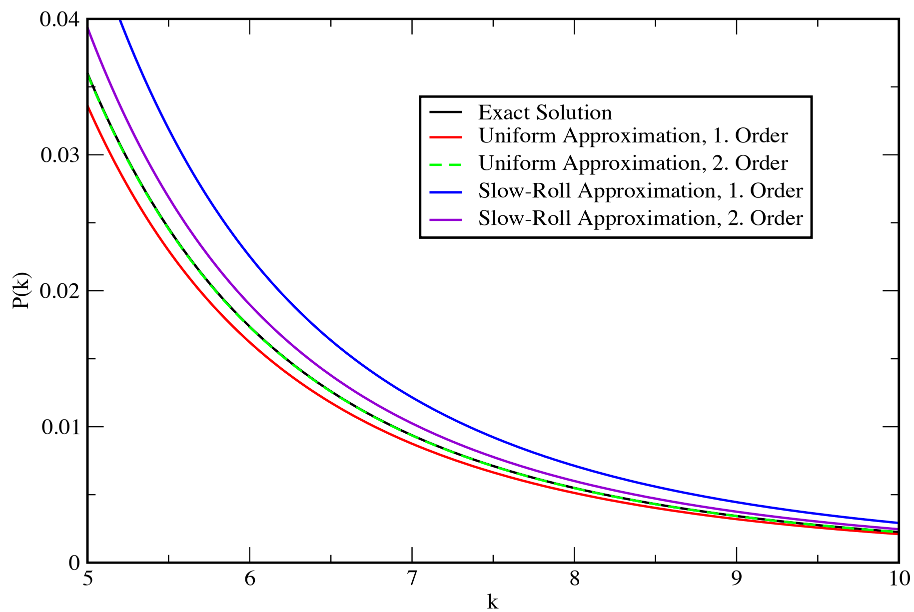
Power Spectrum for Power-Law Inflation

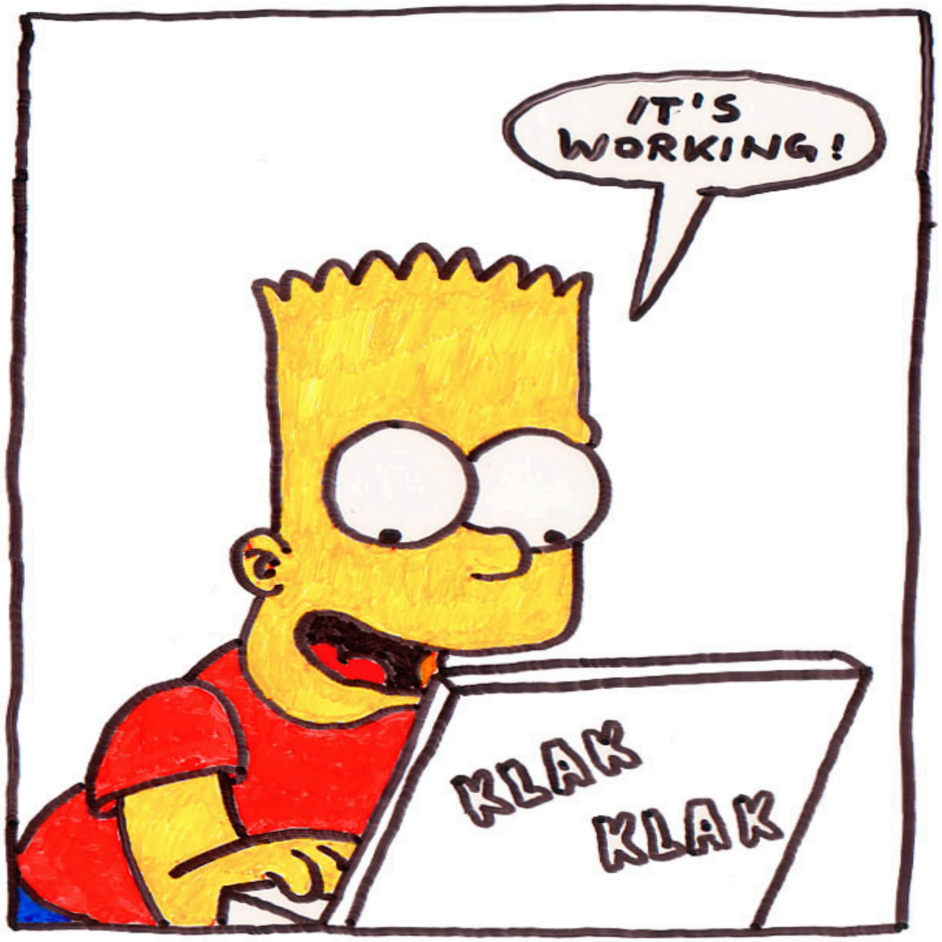
- Expansion parameter evolves like $a(t) \sim t^p \quad \Rightarrow \nu_S = \frac{3}{2} + \frac{1}{1-p}$

Slow-roll expansion: expansion in $\epsilon = \delta = \frac{1}{p}$

$$P_S^{SR}(k) = \frac{H^4}{4\pi^2\dot{\phi}^2}\left[1 - 2(c+1)\frac{1}{p} + (2c^2 + 2c - 5 + \frac{\pi^2}{2})\frac{1}{p^2}\right]$$

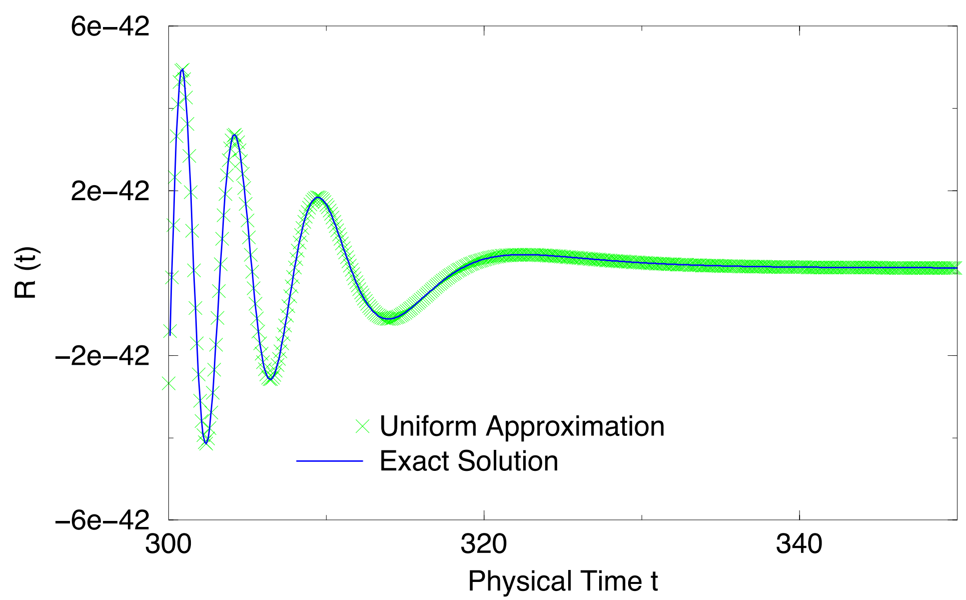
$\Rightarrow$ Slow-roll good for large p, here $p = 2$




IT'S WORKING!
KLAK KLAK

Example in Progress

- Chaotic Inflation, $V(\phi) = m^2\phi^2$
- Important for power spectrum: $R(t) = u_k(t)/z(t)$



Conclusion and Outlook

- New approximation for calculating power spectra and spectral indices
  - Controlled error bounds
  - Systematically improvable

- Generalize error formulae for $\nu \neq$ const.
  - Investigate second order
- Approximation for background equations
  - Numerical examples

- Develop code to generate $n_S(k)$, $n_T(k)$ for arbitrary model $\longrightarrow$ CMBFAST
  - Choose different vacuum states $\longrightarrow$ Trans-Planckian effects
  - Reconstruction of the inflationary equation of state
  - Back reaction problem