Atom Optical Quantum Resonance Dynamics with Gravity

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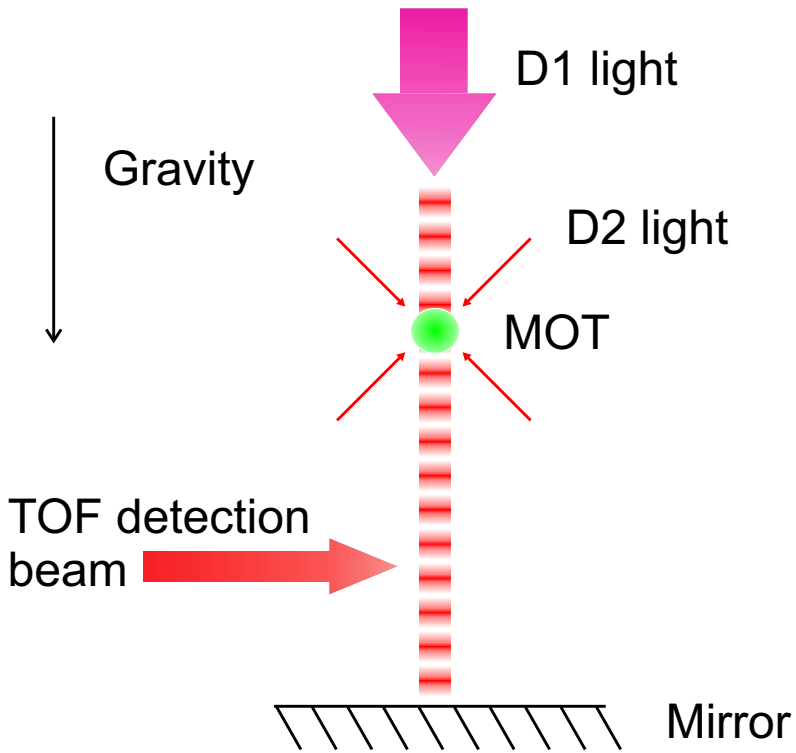
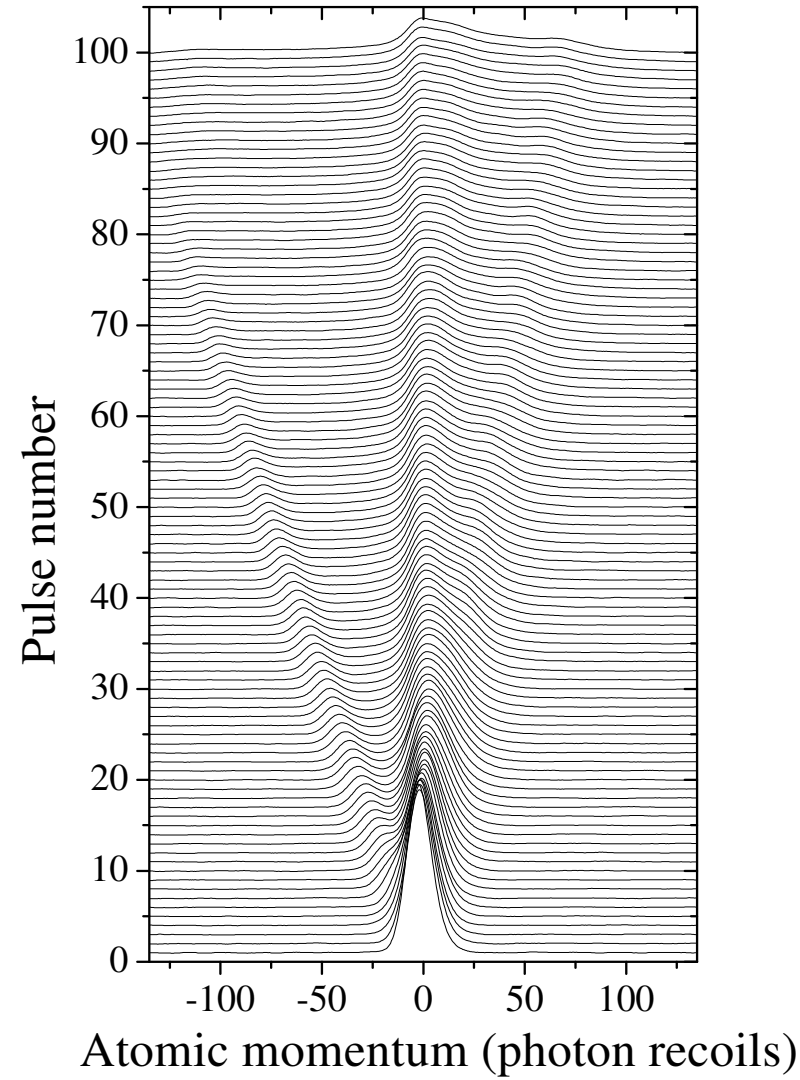
Oberthaler, Godun, d'Arcy, Summy, Burnett, Phys. Rev. Lett. **83**, 4447 (1999)

Observation of Quantum Accelerator Modes





- D1 light far detuned, consider **atomic COM motional dynamics** only.
- Rapid $\sim \delta$-function-like pulses.
- Data displayed in a frame **falling freely with gravity** (pulse periodicity $T = 60.5\ \mu s$).

δ-Kicked Accelerator

- One-dimensional **model** for atomic COM dynamics (**grating recoil $\hbar G = 2$ photon recoils**):

$$\hat{H} = \frac{\hat{p}^2}{2m} + mg\hat{x} - \hbar\phi_d[1 + \cos(G\hat{x})] \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

- In frame **falling freely with gravity**, transforms to

$$\hat{H} = \frac{(\hat{p} - mgt)^2}{2m} - \hbar\phi_d[1 + \cos(G\hat{x})] \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

- Spatially **periodic** in this gauge.

- Invoke **Bloch theory**.

Bloch Theory

- Momentum states **parametrized** by $|p\rangle = |\hbar G(k+\beta)\rangle$, where $k \in \mathbb{Z}$, $\beta \in [0,1)$.

- Position states **parametrized** by $|x\rangle = |G^{-1}(2\pi l + \theta)\rangle$, where $l \in \mathbb{Z}$, $\theta \in [0, 2\pi)$.

- Position and momentum **operators** expanded as

$$G\hat{x} = \left\{ \sum_{l=-\infty}^{\infty} 2\pi l \int_0^{2\pi} d\theta |Gx = 2\pi l + \theta\rangle\langle Gx = 2\pi l + \theta| \right\} = 2\pi \hat{l}$$

$$+ \left\{ \int_0^{2\pi} d\theta\theta \sum_{l=-\infty}^{\infty} |Gx = 2\pi l + \theta\rangle\langle Gx = 2\pi l + \theta| \right\} = \hat{\theta} \textbf{ (angle)}$$

$$\frac{\hat{p}}{\hbar G} = \left\{ \sum_{k=-\infty}^{\infty} k \int_0^1 d\beta |p/\hbar G = k + \beta\rangle\langle p/\hbar G = k + \beta| \right\} = \hat{k}$$

$$+ \left\{ \int_0^1 d\beta\beta \sum_{k=-\infty}^{\infty} |p/\hbar G = k + \beta\rangle\langle p/\hbar G = k + \beta| \right\} = \hat{\beta} \textbf{ (quasimomentum)}$$

Bloch-Wannier Fibration

- Hamiltonian can therefore be written as

$$\hat{H} = \frac{[\hbar G(\hat{k} + \hat{\beta}) - mgt]^2}{2m} - \hbar\phi_d[1 + \cos(\hat{\theta})] \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

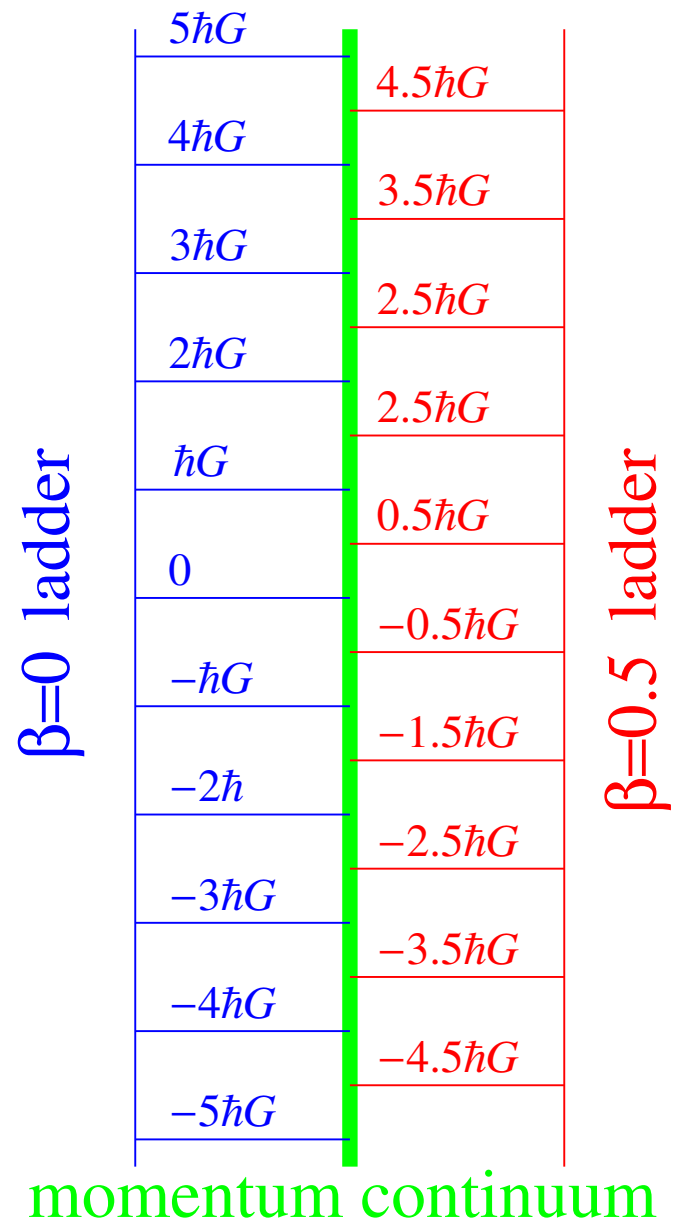
- From commutators

$$[\hat{\beta}, \hat{\theta}] = [\hat{\beta}, \hat{k}] = 0 \Rightarrow [\hat{\beta}, \hat{H}] = 0,$$

  deduce **quasimomentum conserved**.

- Momentum states couple only to other momentum states differing by **integer multiples** of $\hbar G$.

- Separate momentum **'ladders'** for **particular** quasimomenta evolve **independently**.



β-Rotors

- Wavefunctions on particular β ladders are **Bloch states**: $2\pi/G$-periodic functions multiplied by phase factor $e^{i\beta G x}$.

- Equivalently represented by **rotor wavefunctions**.

- Free evolution operator after nth kick [from $t = (n-1)T$ to $t = nT$] for **specific** β

$$\hat{U}_n(\beta) = \exp(-i[(\hbar G^2 T/2m)(\hat{k}+\beta)^2 - gGT^2(n-1/2)(\hat{k}+\beta)]).$$

Talbot Resonance Condition

- Interested when T close to **integer multiples** of **half-Talbot time** $T_{1/2} = 2\pi m/\hbar G^2$.

- Setting $T = (2\pi\ell + \epsilon)m/\hbar G^2$ where $\ell \in \mathbb{Z}$, free evolution operator after nth kick [from $t = (n-1)T$ to $t = nT$] for **specific** β (and $\Omega = gGT^2/2\pi$)

$$\hat{U}_n(\beta) = \exp(-i[(\hat{k}^2/2)(2\pi\ell + \epsilon) + (\hat{k}\beta + \beta^2/2)2\pi T/T_{1/2} - 2\pi\Omega(n - 1/2)\hat{k}]).$$

- Using $\exp(-ik^2\pi\ell) = \exp(-ik\pi\ell)$, ignoring β-dependent phases

$$\hat{U}_n(\beta) = \exp(-i\{\hat{k}\epsilon + 2\pi[\ell/2 + \beta T/T_{1/2} - \Omega(n - 1/2)]\}^2/2\epsilon).$$

Heisenberg Map

- Substituting $\hat{\mathcal{I}} = \hat{k}|\epsilon|$, kick-to-kick **time evolution operator**

$$\hat{F}_n(\beta) = \exp(-i\{\hat{\mathcal{I}} + \mathrm{sgn}(\epsilon)2\pi[\ell/2 + \beta T/T_{1/2} - \Omega(n-1/2)]\}^2/2\epsilon)\exp(iK\cos\hat{\theta}/|\epsilon|),$$

where $K = \phi_d|\epsilon|$, and $[\hat{\theta}, \hat{\mathcal{I}}] = i|\epsilon|$.

- **Heisenberg** picture, kick-to-kick operator evolution (on **specific** β ladder)

$$\begin{aligned}\hat{\theta}(t = [n+1]T) =& \hat{\theta}_{n+1} = \hat{F}_n^\dagger(\beta)\hat{\theta}_n\hat{F}_n(\beta),\\ \hat{\mathcal{I}}(t = [n+1]T) =& \hat{\mathcal{I}}_{n+1} = \hat{F}_n^\dagger(\beta)\hat{\mathcal{I}}_n\hat{F}_n(\beta).\end{aligned}$$

- Corresponding **Heisenberg** map

$$\begin{aligned}\hat{\theta}_{n+1} =& \hat{\theta}_n + \mathrm{sgn}(\epsilon)\hat{\mathcal{J}}_{n+1},\\ \hat{\mathcal{J}}_{n+1} =& \hat{\mathcal{J}}_n - K\sin\hat{\theta}_n - \mathrm{sgn}(\epsilon)2\pi\Omega,\end{aligned}$$

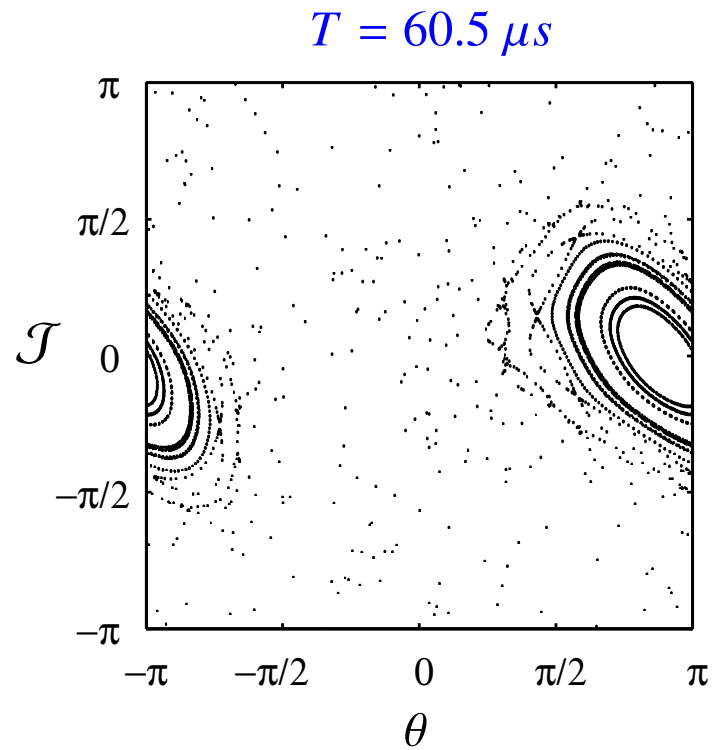
where $\hat{\mathcal{J}}_n = \hat{\mathcal{I}}_n + \mathrm{sgn}(\epsilon)2\pi[\ell/2 + \beta T/T_{1/2} - \Omega(n-1/2)]$, and $[\hat{\theta}, \hat{\mathcal{J}}] = [\hat{\theta}, \hat{\mathcal{I}}] = i|\epsilon|$.

ϵ-Classical Map

- As $T \to \ell T_{1/2}$, commutator $[\hat{\theta}, \hat{\mathcal{J}}] = i|\epsilon| \to 0$, replace operators $\hat{\theta}$, $\hat{\mathcal{J}}$ with their **mean values** θ, $\mathcal{J}$.

- Presence of quantum accelerator modes due to significant **island structures** around **stable periodic orbits** in phase space of **ϵ-classical map:**

$$\begin{aligned}\theta_{n+1} &= \theta_n + \mathrm{sgn}(\epsilon)\mathcal{J}_{n+1},\\ \mathcal{J}_{n+1} &= \mathcal{J}_n - K\sin\theta_n - \mathrm{sgn}(\epsilon)2\pi\Omega.\end{aligned}$$



- Phase space 2π-periodic in $\mathcal{J}$, periodic orbit therefore $\Rightarrow \mathcal{J}_{N\mathfrak{p}} = \mathcal{J}_0 + 2\pi\mathfrak{j}$. Thus,

$$\mathcal{I}_{N\mathfrak{p}} = \mathcal{I}_0 + 2\pi N[\mathfrak{j} + \mathrm{sgn}(\epsilon)\Omega\mathfrak{p}]$$

$\mathfrak{p}$ the **order**, $\mathfrak{j}$ the **jumping index** ($\mathfrak{j}/\mathfrak{p}$ equivalent to **winding number** when phase space wrapped onto torus).

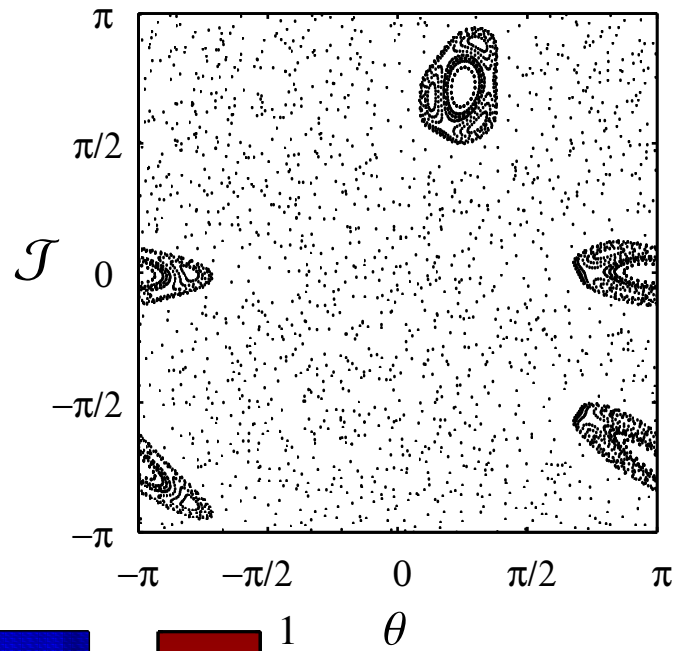
Comparison: Theory and Experiment

- In **falling frame**, accelerator mode momentum therefore predicted theoretically **(white lines)** by:

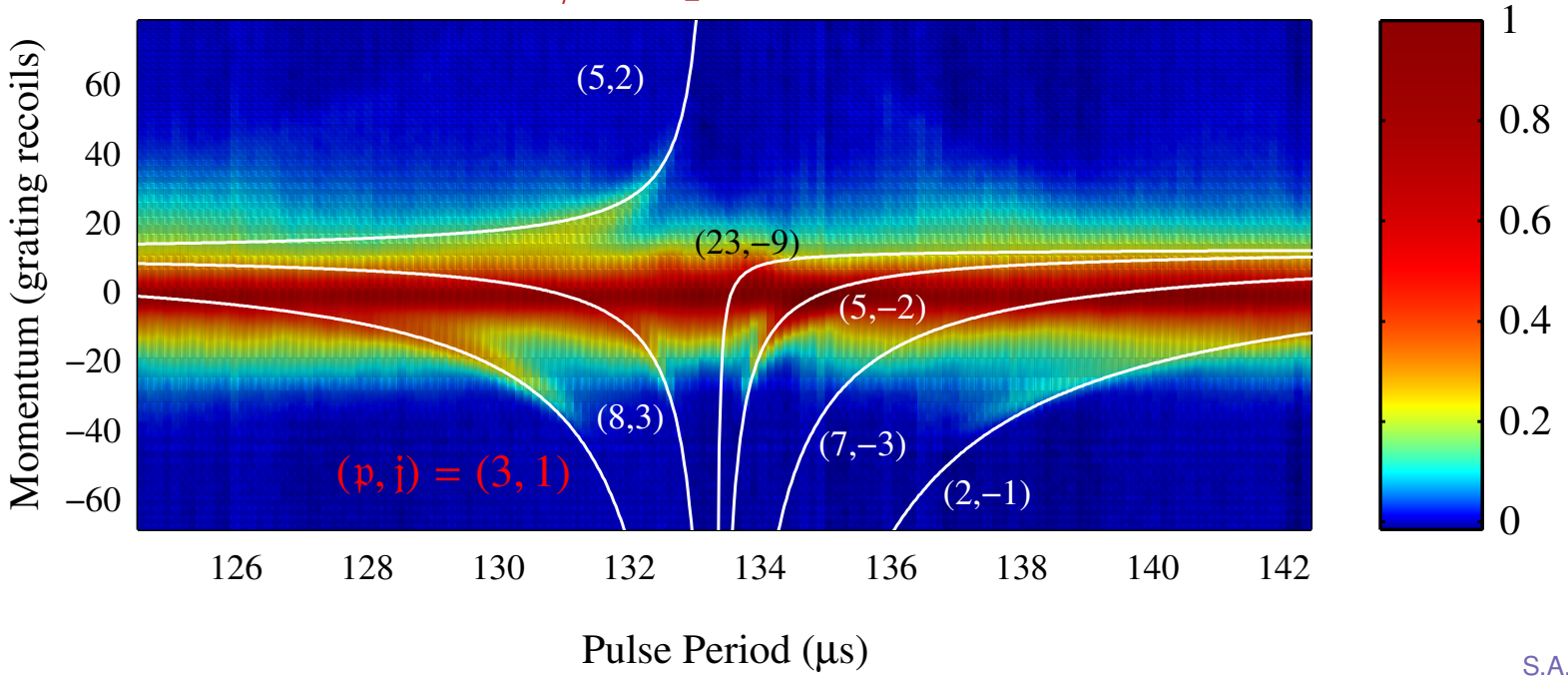
$$p_N \simeq p_0 + \frac{2\pi N}{|\epsilon|}\left[\frac{\mathfrak{j}}{\mathfrak{p}} + \mathrm{sgn}(\epsilon)\frac{gGT^2}{2\pi}\right]\hbar G,$$

independent of mean $\phi_d = K/|\epsilon| \approx 0.8\pi$.

$(\mathfrak{p}, \mathfrak{j}) = (3, 1)$, $T = 130\ \mu s$

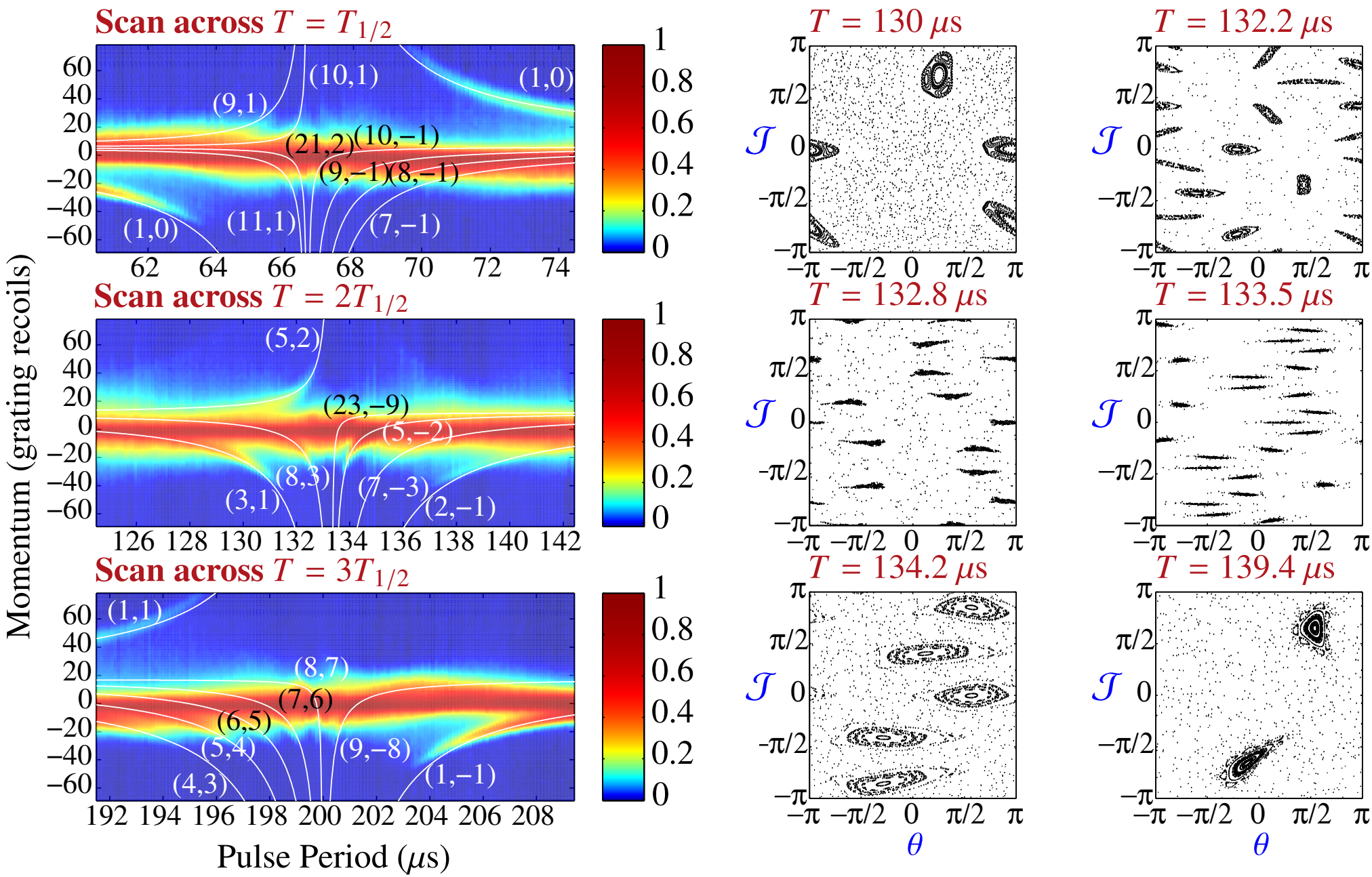


Scan across $T = 2T_{1/2}$, 30 pulses



Schlunk, d'Arcy, Gardiner, Summy, Phys. Rev. Lett. **90**, 124102 (2003)

High-Order Quantum Accelerator Modes



Farey Tree Structure

- Necessary condition for **existence** of periodic orbit $|\mathfrak{j}/\mathfrak{p} + \mathrm{sgn}(\epsilon)\Omega| \leq K/2\pi$; rewrite as

$$-\frac{|\epsilon|}{2\pi}\left[\phi_d + 2\ell(\Omega/T^2)T_{1/2}^2\right] \leq \frac{\mathfrak{j}}{\mathfrak{p}} + \mathrm{sgn}(\epsilon)\ell^2(\Omega/T^2)T_{1/2}^2 \leq \frac{|\epsilon|}{2\pi}\left[\phi_d - 2\ell(\Omega/T^2)T_{1/2}^2\right].$$

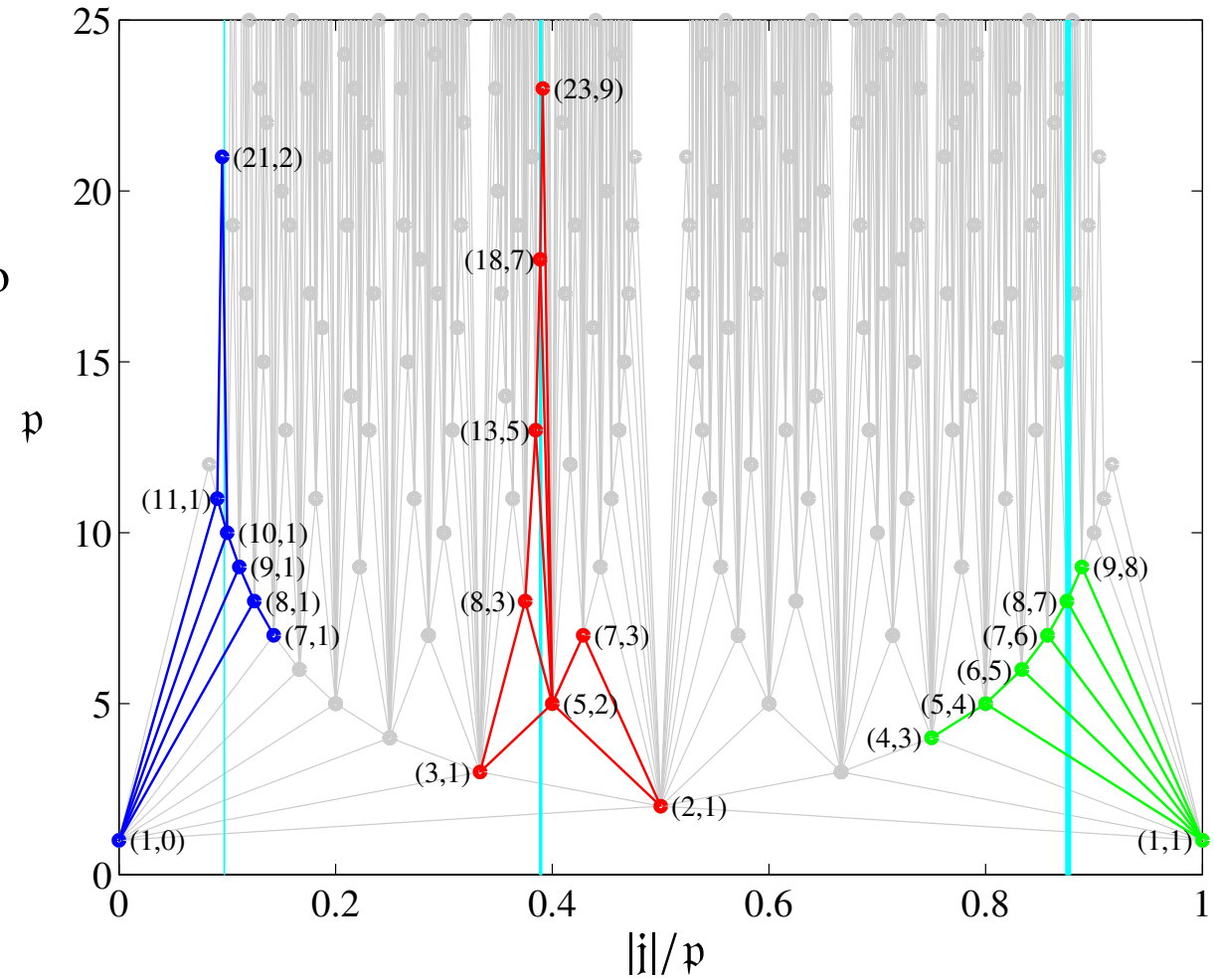
- $\Omega/T^2 = gG/2\pi$ **independent** of T, therefore of ϵ.

- $|\epsilon| \to 0 \Rightarrow |\mathfrak{j}|/\mathfrak{p}$ must converge to $\ell^2(\Omega/T^2)T_{1/2}^2 = \ell^2 g(m/h)^2\lambda_{\mathrm{spat}}^3$

- As $|\epsilon| \to 0$, new QAM often predicted by **Farey rule**,

$$\frac{|\mathfrak{j}_{\mathrm{new}}|}{\mathfrak{p}_{\mathrm{new}}} = \frac{|\mathfrak{j}_+| + |\mathfrak{j}_-|}{\mathfrak{p}_+ + \mathfrak{p}_-}$$

where $\mathfrak{j}_+/\mathfrak{p}_+ > \ell^2(\Omega/T^2)T_{1/2}^2$, $\mathfrak{j}_-/\mathfrak{p}_- < \ell^2(\Omega/T^2)T_{1/2}^2$.



Mode-Locking

- Farey tree structure generic feature of **mode-locking** phenomena.

- Incorporate **damping term** to illustrate connection:

$$\begin{aligned}\mathcal{J}_{n+1} =& e^{-\gamma}\mathcal{J}_n - K\sin(\theta_n) - 2\pi\Omega,\\ \theta_{n+1} =& \theta_n + \mathcal{J}_{n+1}.\end{aligned}$$

- Recast as $\theta_{n+1} = \theta_n + e^{-\gamma(n+1)}\mathcal{J}_0 - \sum_{k=0}^{n} e^{-\gamma(n-k)}[K\sin(\theta_k) + 2\pi\Omega]$.

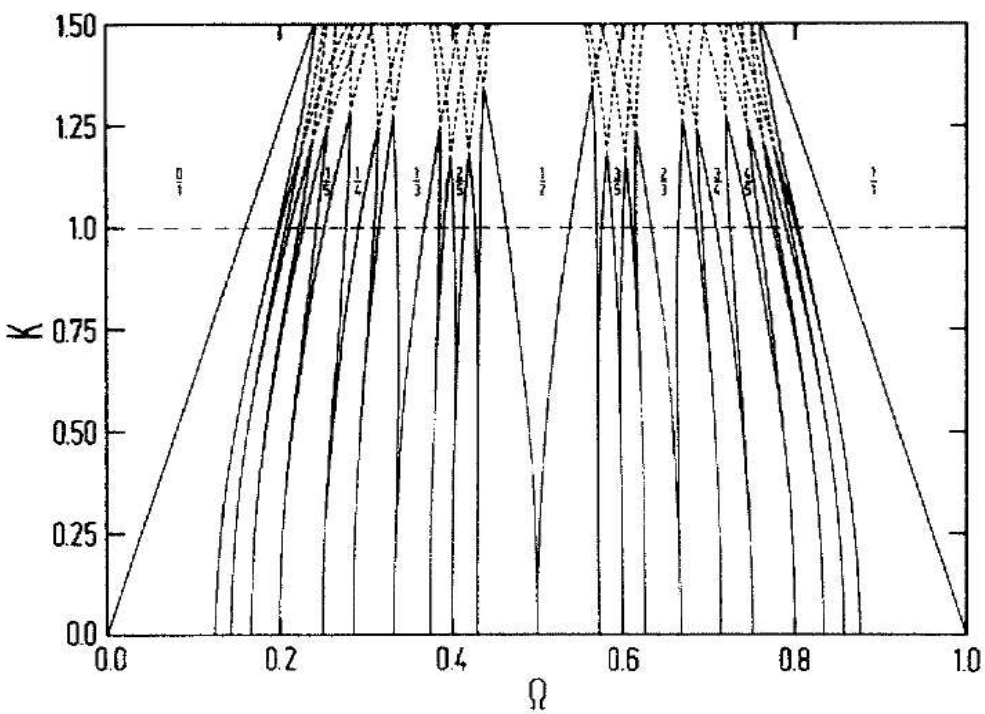
- Finite $\gamma > 0$ leads to **decay** of system memory.
- $\gamma \to \infty$ thus analogous to a **Markov approximation**, produces **sine-circle map**:

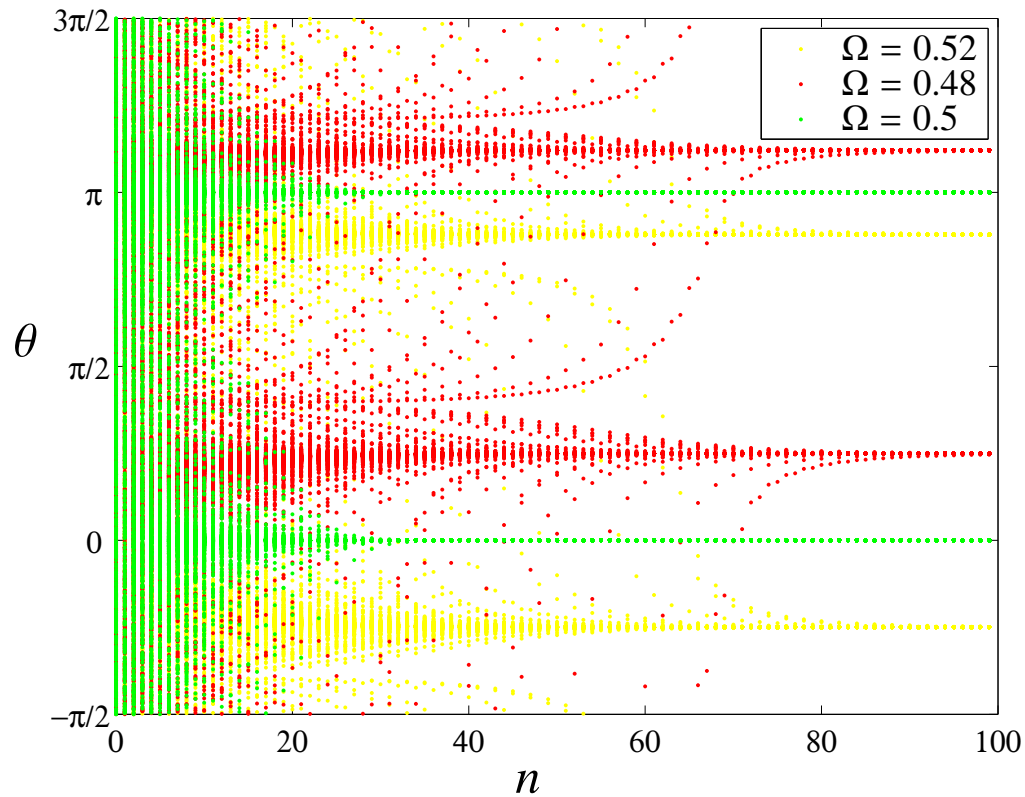
$$\theta_{n+1} = \theta_n - K\sin(\theta_n) - 2\pi\Omega.$$

- Mode-locking **paradigm**, where Ω called **unperturbed winding number**.

Sine-Circle Map

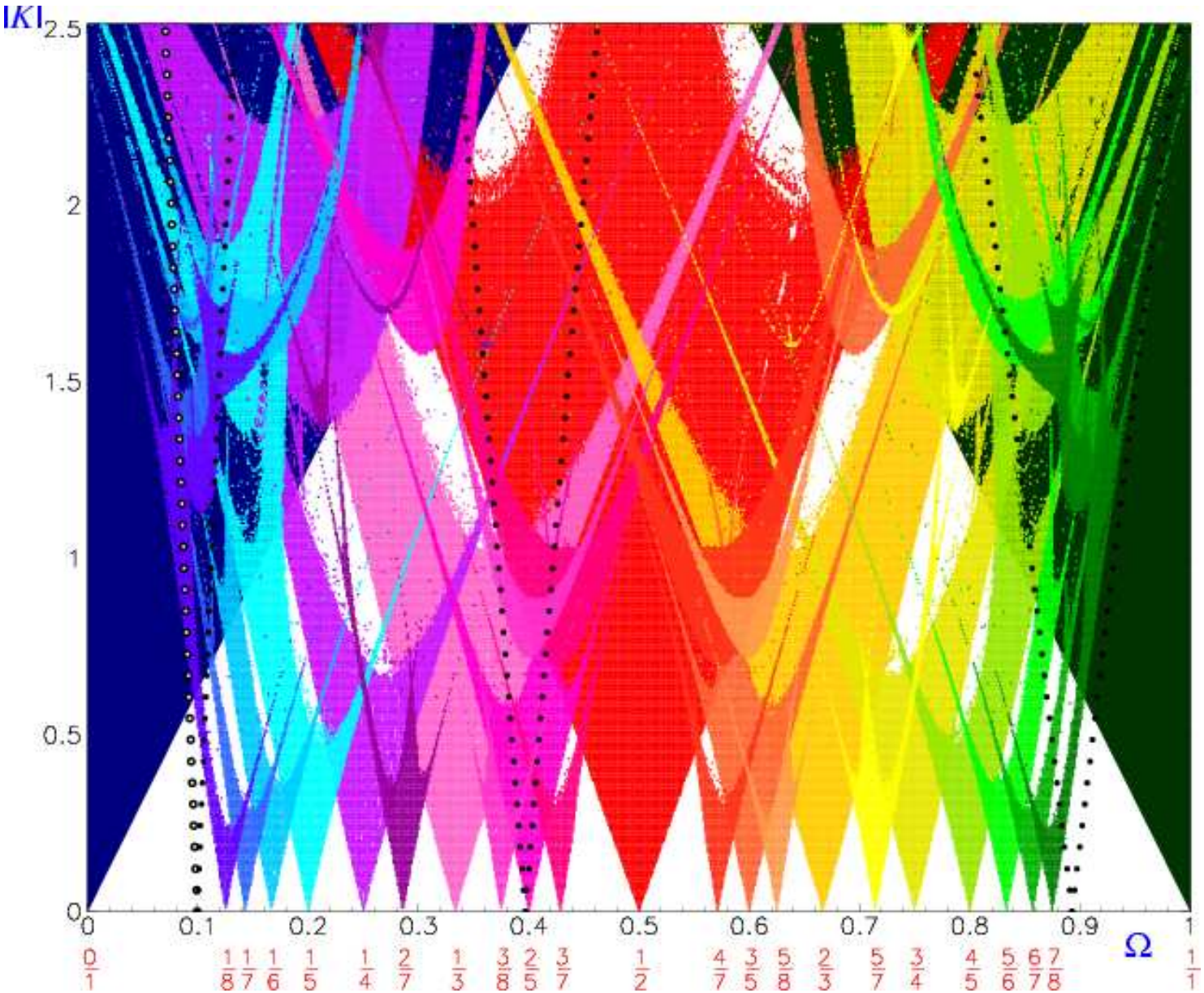
- If $K = 0$ and $\Omega = \mathfrak{j}/\mathfrak{p}$, **all** trajectories **periodic**.
- If $0 < K < 1$, for **finite** range of Ω around $\mathfrak{j}/\mathfrak{p}$, periodic trajectory with winding number $\mathfrak{j}/\mathfrak{p}$ **persists**.
- Attracts **all other orbits** asymptotically in time.





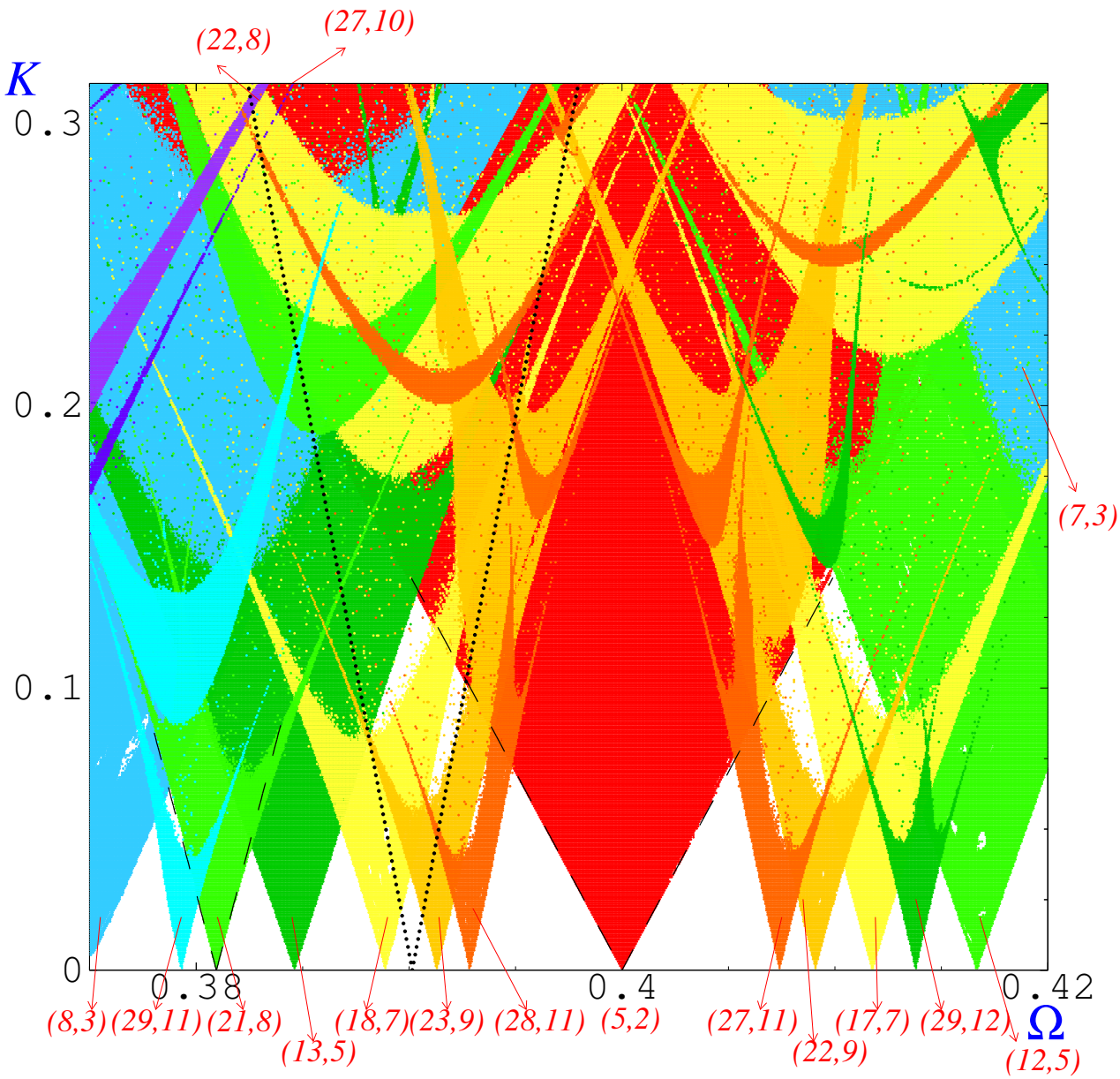
- Widths in Ω of **mode-locking intervals** exponentially small in $\mathfrak{p}$, increase with K up to $K \geq 1$.
- Form regions in (Ω, K) parameter space called **Arnol'd tongues**, terminating at $(\Omega, K) = (\mathfrak{j}/\mathfrak{p}, 0)$.

Non-Dissipative ‘Arnol’d Tongues’



- Regions wherein $(\mathfrak{p}, \mathfrak{j})$ **stable periodic orbits** (identified with Ω near $|\mathfrak{j}|/\mathfrak{p}$) observed numerically.
- Dotted lines mark **loci** of **experimentally explored points** when scanning across $T = T_{1/2}$, $T = 2T_{1/2}$, $T = 3T_{1/2}$.
- **‘Tongues’** may **overlap**; true Arnol’d tongues **repel**.

Properties of ‘Arnol’d Tongues’



- Canonical perturbation theory (K small, Ω close to $\mathfrak{j}/\mathfrak{p}$) reveals condition for **stable periodic orbit**:

$$|K| > 2\pi\sqrt{\mathfrak{p}}\left|\frac{|\mathfrak{j}|}{\mathfrak{p}} - \Omega\right|.$$

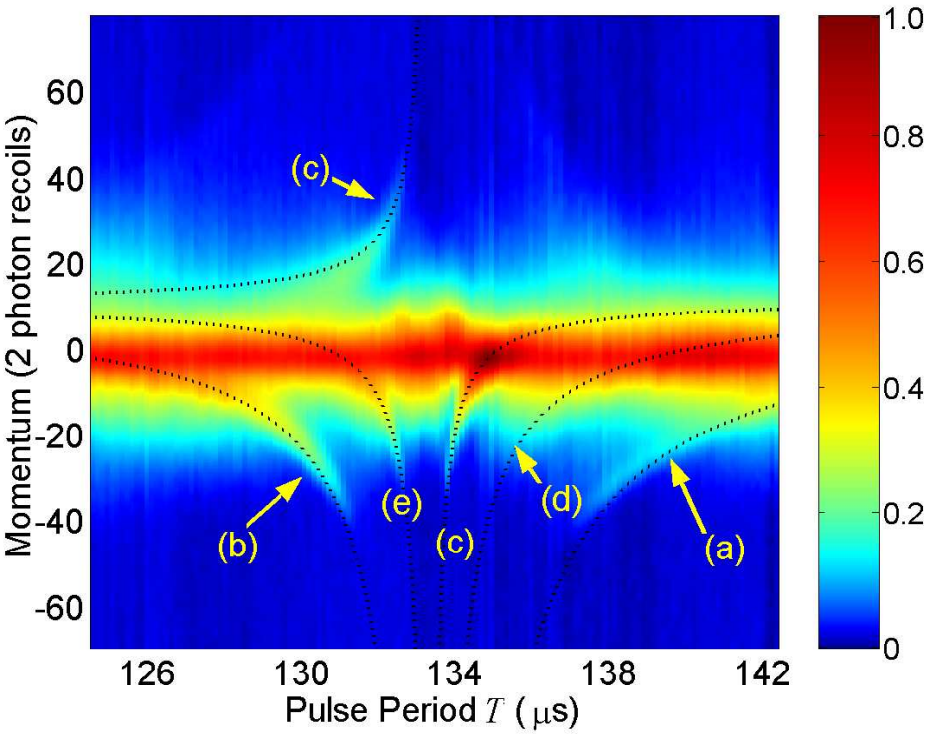
- Numerics and scaling considerations reveal tongue **breaks down** at $K \sim 2\pi\mathfrak{p}^{-3/2}$.

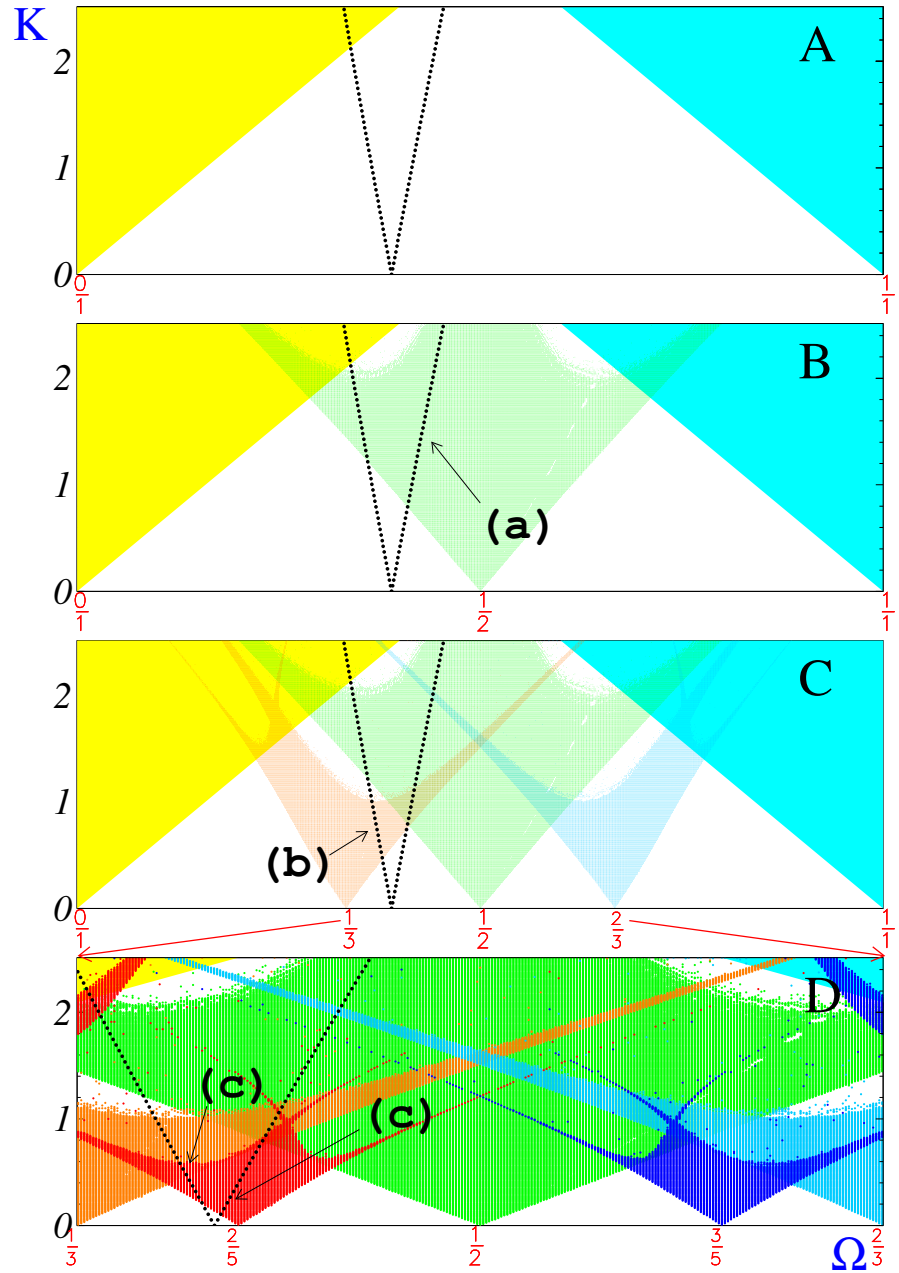
- Combine to form **bound** within which expect to observe QAM **(dashed lines)**.

Correspondence of Arnol'd Tongues to QAM

- As $T \to 2T_{1/2}$, **new QAM** predicted by appearance of **new tongue**, generally as determined by **Farey rule**

- (a) $(\mathfrak{p}, \mathfrak{j}) = (2, -1)$, (b) $(\mathfrak{p}, \mathfrak{j}) = (3, 1)$, (c) $(\mathfrak{p}, \mathfrak{j}) = (5, -2)$, (d) $(\mathfrak{p}, \mathfrak{j}) = (7, -3)$, (e) $(\mathfrak{p}, \mathfrak{j}) = (8, 3)$.





Second-Order Cumulants

- ϵ-classical formalism **highly effective** even if $|\epsilon|$ relatively large (up to ~ 0.8). **Why?**

- Expand **moments** in **means** and **variances**, neglecting higher-order cumulants, e.g.

$$\langle \hat{\theta}^n \rangle = \sum_{k=0}^{\lceil (n-1)/2 \rceil} \frac{n!}{2^k (n-2k)! k!} \theta^{n-2k} (\sigma^2)^k.$$

- In resulting **second-order Cumulant map** must consider each of θ, $\mathcal{J}$, $\sigma^2 = \langle \hat{\theta}^2 \rangle - \theta^2$, $S^2 = \langle \hat{\mathcal{J}}^2 \rangle - \mathcal{J}^2$, $\Upsilon = \langle \hat{\theta}\hat{\mathcal{J}} + \hat{\mathcal{J}}\hat{\theta} \rangle / 2 - \theta \mathcal{J}$, **independently:**

$$\begin{aligned}
\theta_{n+1} &= \theta_n + \mathrm{sgn}(\epsilon) \mathcal{J}_{n+1}, \\
\mathcal{J}_{n+1} &= \mathcal{J}_n - K e^{-\sigma_n^2/2} \sin\theta_n - \mathrm{sgn}(\epsilon) 2\pi\Omega, \\
\sigma_{n+1}^2 &= \sigma_n^2 + S_{n+1}^2 + 2\mathrm{sgn}(\epsilon)(\Upsilon_n - K e^{-\sigma_n^2/2} \sigma_n^2 \cos\theta_n), \\
S_{n+1}^2 &= S_n^2 - 2K e^{-\sigma_n^2/2} \Upsilon_n \cos\theta_n + K^2 (1 - e^{-\sigma_n^2})[1 + e^{-\sigma_n^2} \cos(2\theta_n)]/2, \\
\Upsilon_{n+1} &= \Upsilon_n - K e^{-\sigma_n^2/2} \sigma_n^2 \cos\theta_n + \mathrm{sgn}(\epsilon) S_{n+1}^2.
\end{aligned}$$

- If **variances constant**, θ, $\mathcal{J}$ dynamics same as ϵ-classical map, with $\tilde{K} = e^{-\sigma^2/2} K$.

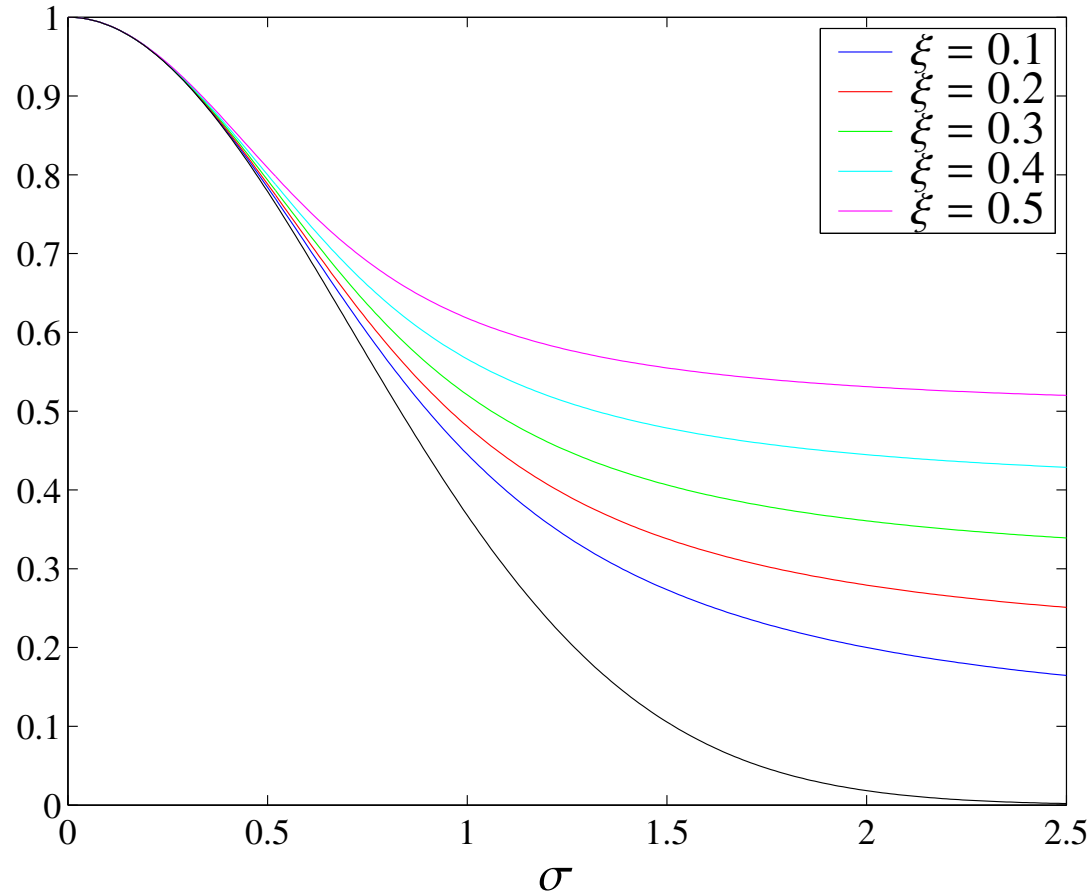
Second-Order Cumulants: Periodic Orbits

- Periodic orbit solutions of second-order cumulant map **do not exist**.
- e.g. $(\mathfrak{p}, \mathfrak{j}) = (1, 0)$ **stable fixed points** must solve

$$\underbrace{e^{-\sigma^2}}_{\text{black line}} = \underbrace{\xi + \sigma^2 + \sqrt{\sigma^4 + (1-\xi)^2}}_{\text{coloured lines}},$$

where
$\xi = (2\pi\Omega/K)^2 = (gGT^2/\phi_d\epsilon)^2$.

- States completely described by **means** and **variances** are **Gaussian**.
- Stable Gaussian solutions exist only for **harmonic oscillator**.



Gaussian Ansatz

- Near **stable** periodic orbits, in ϵ-classical phase space local dynamics **approximately** harmonic.

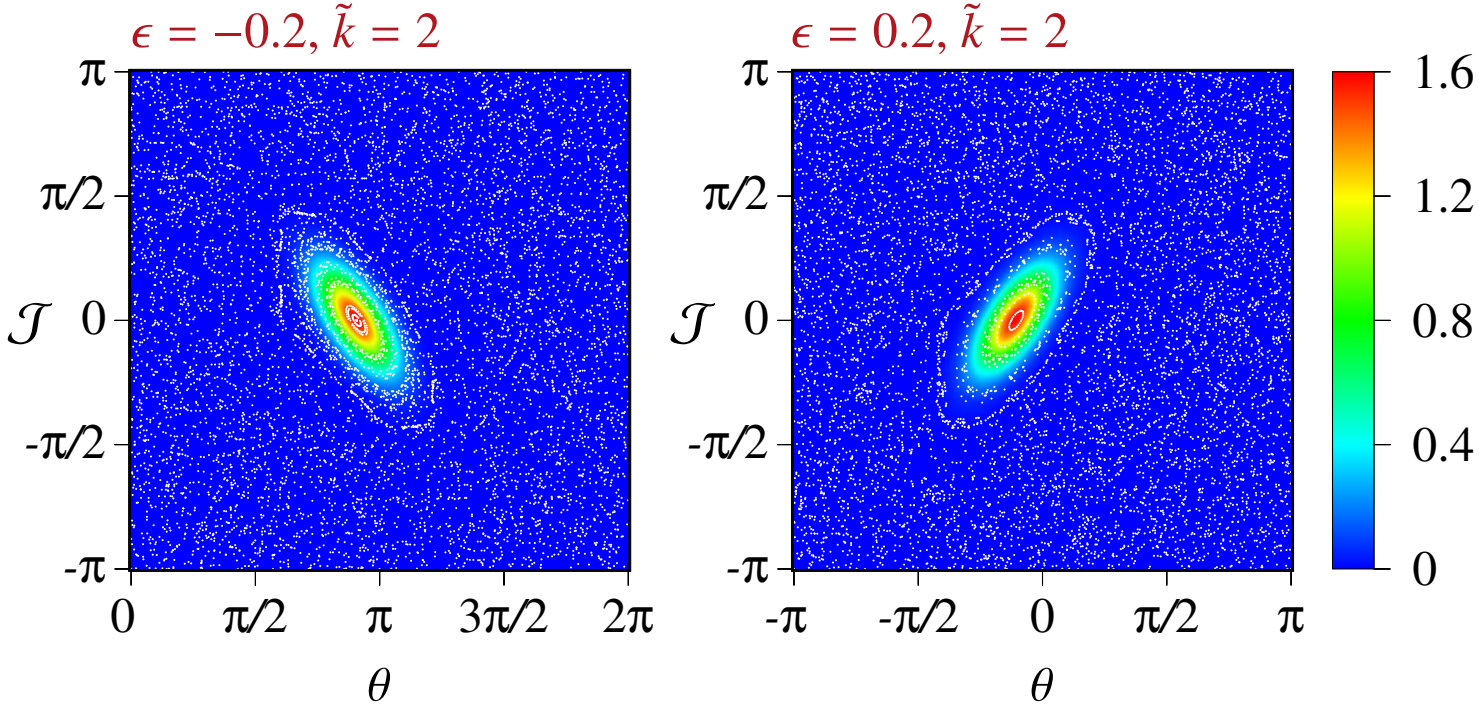
- **Gaussian ansatz** produces **Gaussian map:**

$$
\begin{aligned}
\theta_{n+1} &= \theta_n + \mathrm{sgn}(\epsilon)\mathcal{J}_{n+1}, \\
\mathcal{J}_{n+1} &= \mathcal{J}_n - Ke^{-\sigma_n^2/2}\sin\theta_n - \mathrm{sgn}(\epsilon)2\pi\Omega, \\
\sigma_{n+1}^2 &= \sigma_n^2 + 2\mathrm{sgn}(\epsilon)(\Upsilon_n - Ke^{-\sigma_n^2/2}\sigma_n^2\cos\theta_n) + [2(\Upsilon_n - Ke^{-\sigma_n^2/2}\sigma_n^2\cos\theta_n)^2 + \epsilon^2]/4\sigma_n^2, \\
\Upsilon_{n+1} &= \Upsilon_n - Ke^{-\sigma_n^2/2}\sigma_n^2\cos\theta_n + \mathrm{sgn}(\epsilon)[2(\Upsilon_n - Ke^{-\sigma_n^2/2}\sigma_n^2\cos\theta_n)^2 + \epsilon^2]/4\sigma_n^2,
\end{aligned}
$$

with S^2 deduced from $\sigma^2 S^2 - \Upsilon^2 = \epsilon^2/4$.

- **'Stable'** Gaussian periodic orbits possible.

Gaussian Ansatz: Fixed Points

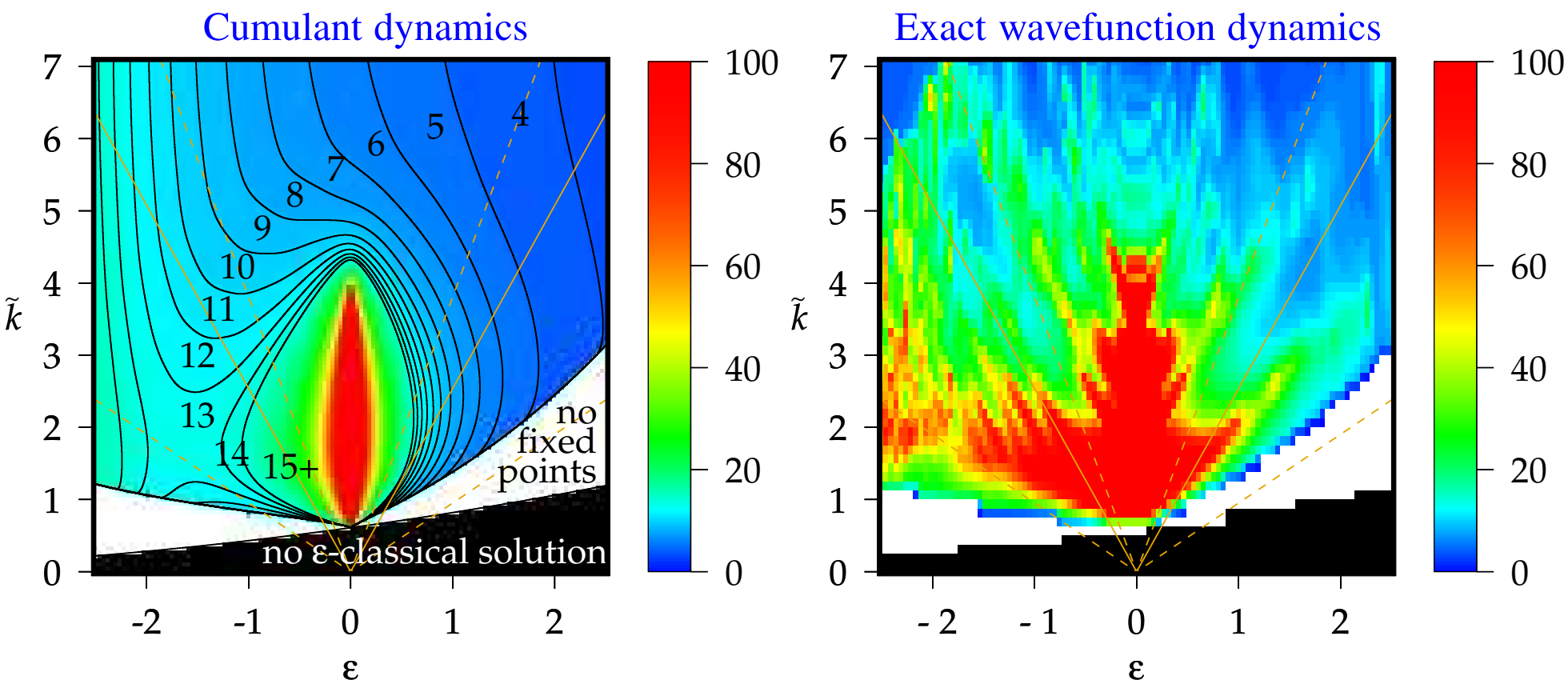


- Exist **'stable'** $(\mathfrak{p}, \mathfrak{j}) = (1, 0)$ Gaussian solutions where $\theta_{n+1} = \theta_n$, $\mathcal{J}_{n+1} = \mathcal{J}_n$, $\sigma^2_{n+1} = \sigma^2_n$, $\Upsilon^2_{n+1} = \Upsilon_n$.

- Compare with ϵ-classical phase space by plotting corresponding **Wigner functions**.

- Solutions stable on **finite timescale**, during which ϵ-classical map with $\tilde{K} = e^{-\sigma^2/2}K$ should describe θ, $\mathcal{J}$ QAM dynamics.

Cumulant and Wavefunction Dynamics

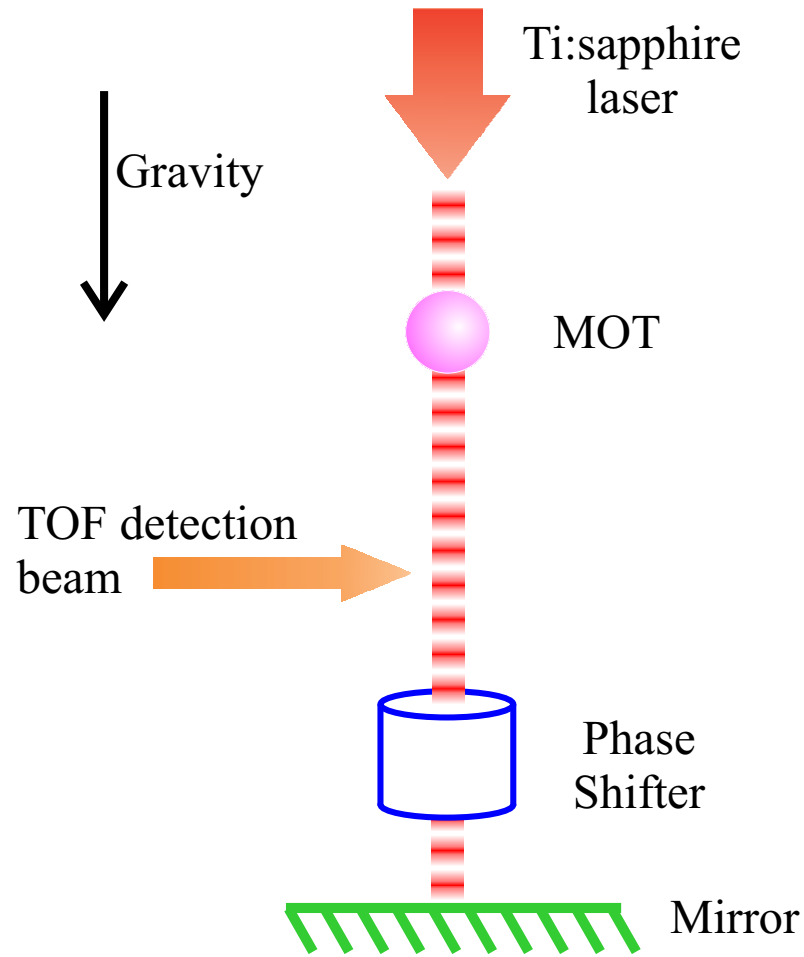


- Propagate **'stable fixed point'** of map for Gaussian ansatz with **second-order cumulant map** and **exact wavefunction dynamics**.

- **Stability** of quantum accelerator modes measured by **number of iterations** for which $|\mathcal{J}| < \pi$.

Tuning 'Gravitational Acceleration'

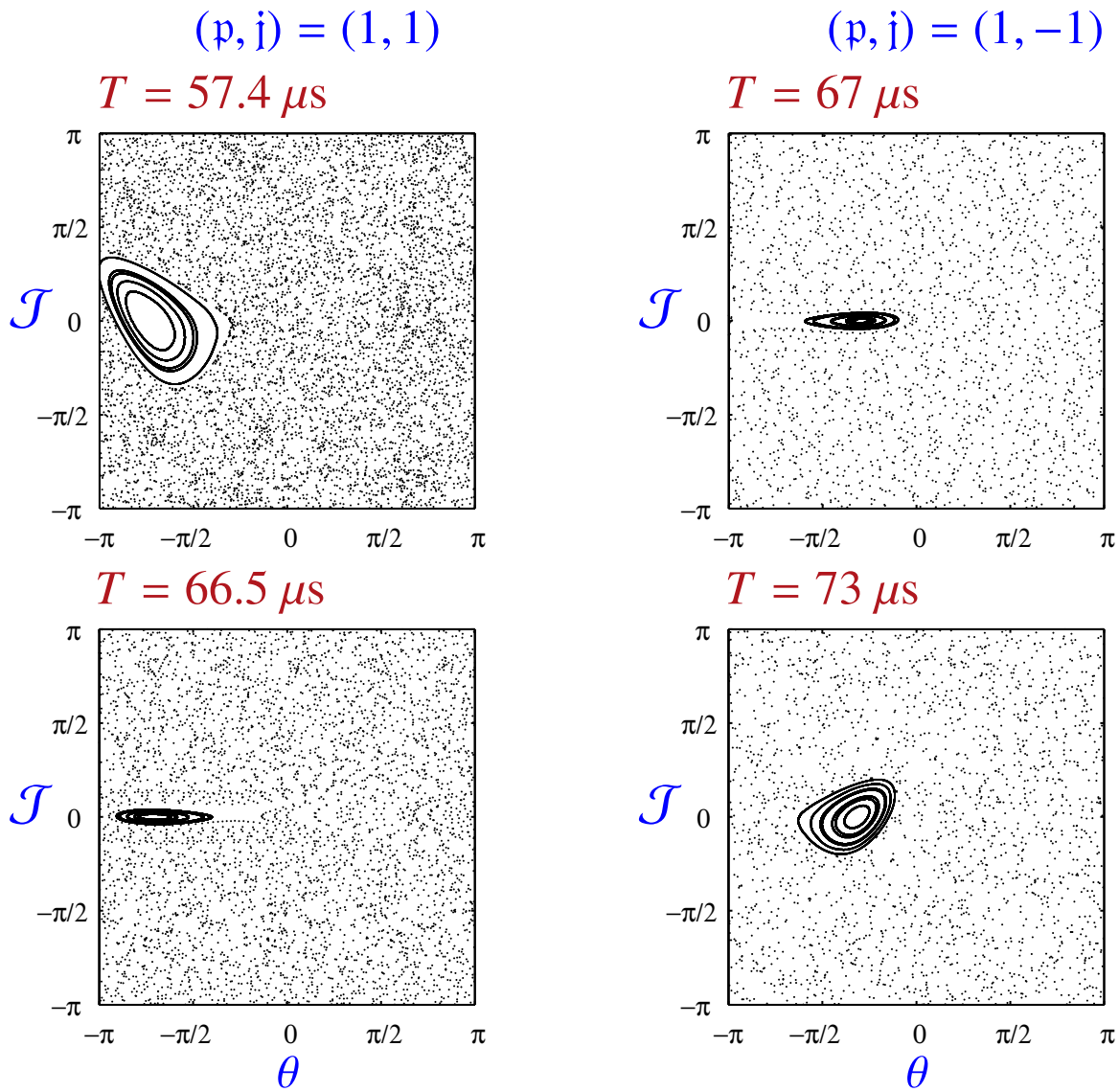
- Insert voltage controlled **crystal phase modulator** between mirror and atomic cloud.
- Used to **stroboscopically accelerate** standing wave profile.
- Possible to effectively **modify** g.
- If set $g = (r/s)(h/m)^2/\lambda_{\text{spat}}^3$, $r/s \in \mathbb{Q}$ then **necessary condition** for existence of periodic orbit:



$$-\frac{|\epsilon|}{2\pi}\left[\phi_d + 2\ell(\Omega/T^2)T_{1/2}^2\right] \leq \frac{\mathfrak{j}}{\mathfrak{p}} + \text{sgn}(\epsilon)\frac{r}{s} \leq \frac{|\epsilon|}{2\pi}\left[\phi_d - 2\ell(\Omega/T^2)T_{1/2}^2\right].$$

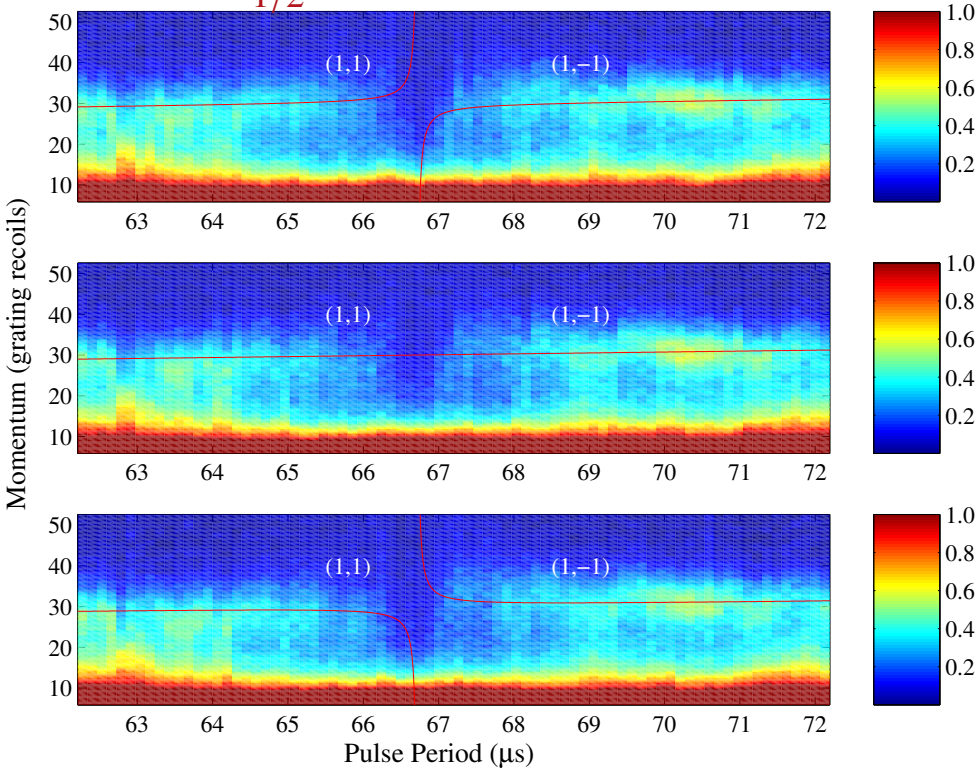
Example: $r/s = 1/1$

- As $\mathfrak{p}/\mathfrak{j} + \mathrm{sgn}(\epsilon) r/s$ can $= 0$, structure of ϵ-classical phase space **dramatically simplified**
- **Vertex** of locus of **experimentally explored points** coincides with vertex of an **Arnol'd tongue**.
- Locus entirely **inside** (or potentially outside) tongue.
- Have **pairs** of $(\mathfrak{p}, \mathfrak{j}) = (r, -\mathrm{sgn}(\epsilon) s)$ QAM on either side of $T = T_{1/2}$.

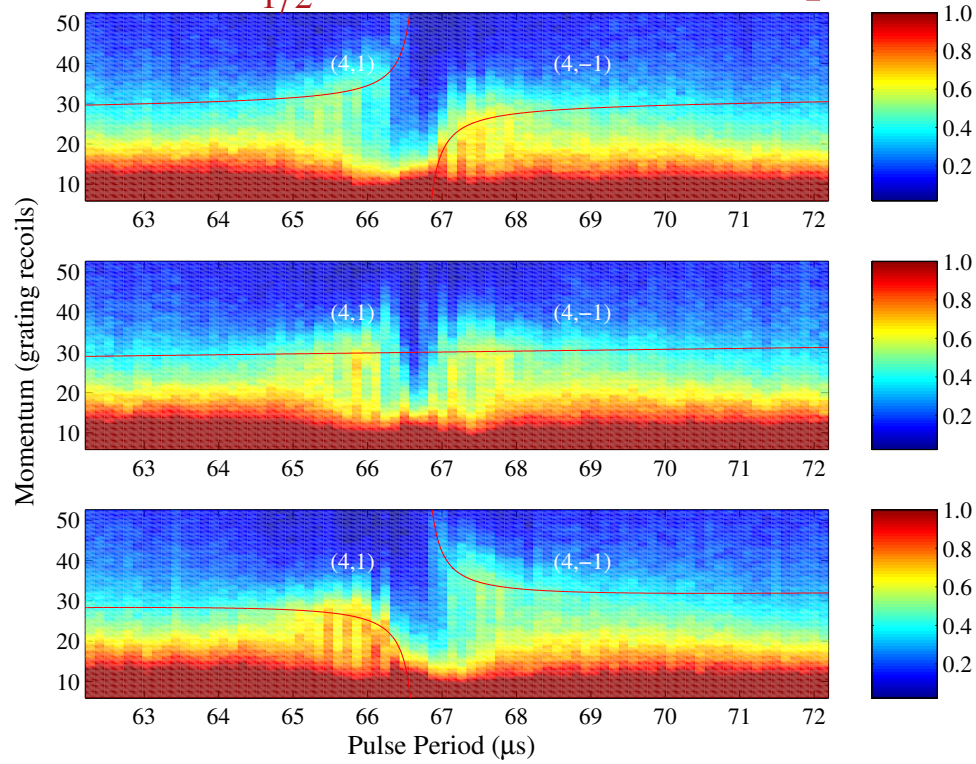


Observation of Gravitational Resonances

$(\Omega/T^2)T_{1/2}^2 = 1/1 \pm 8.5 \times 10^{-4}$ **(15 pulses)**



$(\Omega/T^2)T_{1/2}^2 = 1/4 \pm 8.5 \times 10^{-4}$ **(60 pulses)**



- Consider momentum of orbits where $\mathfrak{p}/\mathfrak{j} = -\mathrm{sgn}(\epsilon) r/s$

- Letting $(\Omega/T^2)T_{1/2}^2 = r/s + w$: then $p_N \simeq p_0 + N\left[\frac{r}{s}\left(2 + \frac{\epsilon}{2\pi}\right) + w\left(\frac{2\pi}{\epsilon} + 2 + \frac{\epsilon}{2\pi}\right)\right]\hbar G.$

Relationship of g to h/m

- **Linear** with ϵ behaviour of QAM when $g = (h/m)^2(r/s)/\lambda_{\text{spat}}^3$.
- Determines **total** acceleration (sinusoidal potential plus gravitational).
- Subtract **imposed acceleration** to determine relationship of **gravitational acceleration** to h/m.
- **Our setup:** imposed acceleration calibrated to within $\sim 1\ \%$.
- **Two counterpropagating waves setup:** calibration to between 1 ppm and 1 ppb possible.

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- **Recent Publications:**

  physics/0311045 [Ma, d'Arcy, Gardiner]

  physics/0310143 [Bach, d'Arcy, Gardiner, Burnett]

  Phys. Rev. Lett. **90**, 124102 (2003) [Schlunk, d'Arcy, Gardiner, Summy]