14 Anomalies and the NSVZ β function

14.1 Fujikawa's Derivation of the ABJ Anomaly

Consider

$$S = \int d^4x i\psi^\dagger \overline{\sigma}^\mu D_\mu \psi \tag{14.1}$$

Under

$$\psi \to e^{i\alpha(x)}\psi \tag{14.2}$$

$$S \to S - \int d^4x \alpha(x) \partial_\mu(\psi^\dagger \overline{\sigma}^\mu \psi) \tag{14.3}$$

Classically

$$\partial_\mu(\psi^\dagger \overline{\sigma}^\mu \psi) = \partial_\mu j^\mu = 0 \tag{14.4}$$

Quantum Mechanically this is not true. One way to see this is from Fujikawa's path integral derivation of the anomaly. Define

$$\not{D} \equiv i\overline{\sigma}^\mu(\partial_\mu + iA_\mu) \tag{14.5}$$

$$\overline{\not{D}} \equiv i\sigma^\mu(\partial_\mu - iA_\mu) \tag{14.6}$$

Then

$$\int d^4x \psi^\dagger \not{D} \psi = \int d^4x \psi \overline{\not{D}} \psi^\dagger \tag{14.7}$$

and

$$D^2 f_n = \overline{\not{D}}\not{D} f_n = -\lambda_n^2 f_n \tag{14.8}$$

$$\overline{D}^2 g_n = \not{D}\overline{\not{D}} g_n = -\lambda_n^2 g_n \tag{14.9}$$

$$\not{D} f_n = \lambda_n g_n \tag{14.10}$$

$$\overline{\not{D}} g_n = -\lambda_n f_n \tag{14.11}$$

$$\sum_n f_n^*(x) f_n(y) = \delta(x-y) \tag{14.12}$$

$$Tr \int d^4x f_n^*(x) f_m(x) = \delta_{nm} \tag{14.13}$$

In general D^2 and $\overline{D}^2$ have a different number of zero eignevalues. We can expand fermion fields in this basis:

$$\psi(x) = \sum_n a_f f_n(x) \tag{14.14}$$

$$\psi^\dagger(x) = \sum_n b_n g_n(x) \tag{14.15}$$

The path integral measure is

$$\int \mathcal{D}\psi\mathcal{D}\psi^\dagger = \int \Pi_{nm} da_n db_m \tag{14.16}$$

Under a chiral rotation

$$a_n \to a'_n = C_{nm} a_m \tag{14.17}$$
$$b_n \to b'_n = \overline{C}_{nm} b_m \tag{14.18}$$
$$C_{nm} = \mathrm{Tr} \int d^4x e^{i\alpha(x)} f_n^*(x) f_m(x) \tag{14.19}$$
$$\overline{C}_{nm} = \mathrm{Tr} \int d^4x e^{-i\alpha(x)} g_m^*(x) g_n(x) \tag{14.20}$$

So

$$\Pi_{nm} da_n db_m \to (\mathrm{det}C\mathrm{det}\overline{C})^{-1} \Pi_{nm} da_n db_m \tag{14.21}$$
$$(\mathrm{det}C\overline{C})^{-1} = \exp\left(-i\int d^4x\alpha(x)A(x)\right) \tag{14.22}$$
$$A(x) = \mathrm{Tr}\sum_n \left(f_n^*(x)f_n(x) - g_n^*(x)g_n(x)\right) \tag{14.23}$$
$$0 = \frac{\delta Z}{\delta\alpha}|_\alpha = 0 = \langle \partial_\mu j^\mu(x) - iA(x)\rangle \tag{14.24}$$

To evaluate A, we use a smooth regulator function $R(z)$ (like e^{-z})

$$\begin{aligned} A(x) &= \lim_{M\to\infty} \mathrm{Tr}\sum_n R(\lambda_n^2/M^2)\left(f_n^*(x)f_n(x) - g_n^*(x)g_n(x)\right) \\ &= \lim_{M\to\infty} \mathrm{Tr}\sum_n \left(f_n^*(x)R(-D^2/M^2)f_n(x) - g_n^*(x)R(-\overline{D}^2/M^2)g_n(x)\right) \\ &= \lim_{\substack{y\to x \\ M\to\infty}} \mathrm{Tr}\left(R(-D^2/M^2) - R(-\overline{D}^2/M^2)\right)\delta(y-x) \end{aligned} \tag{14.25}$$

Using

$$D^2 = \partial^2 + A^2 - \sigma^{\mu\nu}(F_{\mu\nu} - 2A_{[\mu}\partial_{\nu]}) \tag{14.26}$$
$$\overline{D}^2 = \partial^2 + A^2 + \overline{\sigma}^{\mu\nu}(F_{\mu\nu} - 2A_{[\mu}\partial_{\nu]}) \tag{14.27}$$
$$\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu) \tag{14.28}$$

One finds

$$
\begin{aligned}
A(x) &= \lim_{M \to \infty} \operatorname{Tr}\left(R(-D^2/M^2) - R(-\overline{D}^2/M^2)\right)\delta(y-x) \\
&= \lim_{M \to \infty} \operatorname{Tr}\int \frac{d^4p}{(2\pi)^4}\sum_{n=0}^{\infty}\frac{1}{n!} \\
&\quad \left[\left(\frac{F + a^\mu p_\mu}{M^2}\right)^n - \left(\frac{-\overline{F} - \overline{a}^\mu p_\mu}{M^2}\right)^n\right] R^{(n)}(\frac{p^2 - A^2}{M^2}) \\
&= \frac{1}{16\pi^2}\int_0^\infty x dx R^{(2)}(x)\epsilon^{\mu\nu\alpha\beta}\operatorname{Tr}F_{\mu\nu}F_{\alpha\beta} \\
&= \frac{1}{32\pi^2}F^a_{\mu\nu}\widetilde{F}^a_{\mu\nu}
\end{aligned}
\tag{14.29}
$$

So

$$
\partial_\mu j^\mu(x_E) = \frac{i}{32\pi^2}F^{a\mu\nu}\widetilde{F}^a_{\mu\nu} \tag{14.30}
$$

Integrating A one finds that the integral is related to the number of zero-modes, so

$$
\begin{aligned}
n_\psi - n_{\psi^\dagger} &= \frac{1}{32\pi^2}F^a_{\mu\nu}\widetilde{F}^a_{\mu\nu} \\
&= n
\end{aligned}
\tag{14.31}
$$

14.2 Gauge Anomalies

There are also anomalies for three-point of gauge bosons which are proportional to

$$
\operatorname{Tr}[T^a\{T^b, T^c\}] = A^{abc} \tag{14.32}
$$

which are potentially non-vanishing for $U(1)$ and $SU(N)$ for $N \geq 3$.

$$
A^{abc}(R_1 \oplus R_2) = A^{abc}(R_1) + A^{abc}(R_2) \tag{14.33}
$$

$$
A^{abc}(R_1 \otimes R_2) = \dim(R_1)A^{abc}(R_2) + \dim(R_2)A^{abc}(R_1) \tag{14.34}
$$

For the fundamental (defining) representation we define

$$
d^{abc} = \operatorname{Tr}[T^a_F\{T^b_F, T^c_F\}] \tag{14.35}
$$

So

$$
A^{abc}(R) = A(R)d^{abc} \tag{14.36}
$$

So the gauge anomaly for a theory vanishes if

$$\sum_i A(R_i) = 0 \tag{14.37}$$

The dimension, index and anomaly coefficient of the smallest $SU(N)$ representations are listed below.

Irrep	dim(r)	$2T(r)$	$A(r)$
$\ydiagram{1}$	N	1	1
Adj	N^2-1	$2N$	0
$\ydiagram{1,1}$	$\frac{N(N-1)}{2}$	$N-2$	$N-4$
$\ydiagram{2}$	$\frac{N(N+1)}{2}$	$N+2$	$N+4$
$\ydiagram{1,1,1}$	$\frac{N(N-1)(N-2)}{6}$	$\frac{(N-3)(N-2)}{2}$	$\frac{(N-3)(N-6)}{2}$
$\ydiagram{3}$	$\frac{N(N+1)(N+2)}{6}$	$\frac{(N+2)(N+3)}{2}$	$\frac{(N+3)(N+6)}{2}$
$\ydiagram{2,1}$	$\frac{N(N-1)(N+1)}{3}$	N^2-3	N^2-9
$\ydiagram{2,2}$	$\frac{N^2(N+1)(N-1)}{12}$	$\frac{N(N-2)(N+2)}{3}$	$\frac{N(N-4)(N+4)}{3}$
$\ydiagram{4}$	$\frac{N(N+1)(N+2)(N+3)}{24}$	$\frac{(N+2)(N+3)(N+4)}{6}$	$\frac{(N+3)(N+4)(N+8)}{6}$
$\ydiagram{2,1,1}$	$\frac{N(N+1)(N-1)(N-2)}{8}$	$\frac{(N-2)(N^2-N-4)}{2}$	$\frac{(N-4)(N^2-N-8)}{2}$

If the gauge anomaly does not vanish, then the theory can only make sense as a spontanteously broken theory. Two triangle graphs back to back generate a mass for the gauge bosons. Alternatively if we start with an anomaly free gauge theory and give masses to some subset of the fields so that the anomaly no longer cancels, then the low energy effective theory has an anomaly, but we can only produce such masses if the gauge symmetry is spontaneously broken.

14.3 'tHooft's Anomaly Matching

Consider an asymptotically free gauge theory, with a global symmetry group G. We can easily compute the anomaly for three global G currents in the ultraviolet by looking at triangle diagrams of the fermions. We will call the result A^{UV}. Now imagine that we weakly gauge G with a gauge coupling $g \ll 1$. If $A^{UV} \neq 0$, then we can add some spectators that only have G gauge couplings, they can be chosen such that their G anomaly is $A^S = -A^{UV}$, so the total G anomaly vanishes. Now construct the effective theory at a scale less than the strong interaction scale. If we compute the G anomaly at this scale we add up the triangle diagrams of light fermions, which consist of the spectators and strongly interacting or composite fermions. If G is not broken by the strong interactions its anomaly must still vanish, so

$$0 = A^{IR} + A^{S} \tag{14.38}$$

Thus we have

$$A^{IR} = A^{UV} \tag{14.39}$$

Taking $g \to 0$ changes nothing.

14.4 β Function Mysteries

Three seemingly contradictory statements can be found in the literature:

- the SUSY gauge coupling runs only at one loop

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(3T(Ad) - \sum_j T(r_j)\right) \tag{14.40}$$

  with matter fields Q_j in represenstations r_j

- Novikov, Shifman, Vainshtein, and Zakharov have derived the "exact" β function

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{\left(3T(Ad) - \sum_j T(r_j)(1-\gamma_j)\right)}{1 - T(Ad)\frac{g^2}{8\pi^2}} \tag{14.41}$$

  (where γ_j is the anomalous dimension of the matter field Q_j) and this has been verified up to two loops

- the one and two loop terms in the β function are scheme independent

We have already seen that the first statement is true for the holomorphic coupling. The last statement is easy to prove. Changing renormalization schemes amounts to defining a new coupling

$$g' = g + ag^3 + \mathcal{O}(g^5) \tag{14.42}$$

If

$$\beta(g) = b_1 g^3 + b_2 g^5 + \mathcal{O}(g^7) \tag{14.43}$$

then we simply find

$$\begin{aligned}\beta'(g') &= \beta(g)\frac{\partial g}{\partial g'} \\ &= b_1 g'^3 + b_2 g'^5 + \mathcal{O}(g'^7) \qquad (14.44)\end{aligned}$$

Recall that the holomorphic coupling for pure SUSY Yang-Mills was defined by

$$\mathcal{L}_h = \frac{1}{4}\int d^2\theta \frac{1}{g_h^2} W^a(V_h) W^a(V_h) + h.c. \qquad (14.45)$$

where

$$\frac{1}{g_h^2} = \frac{1}{g^2} - i\frac{\theta_{\mathrm{YM}}}{8\pi^2} = \frac{tau}{4\pi i} \qquad (14.46)$$

$$V_h = (A_\mu^a, \lambda^a, D^a) \qquad (14.47)$$

We could also define a gauge coupling for canonically normalized fields

$$\mathcal{L}_c = \frac{1}{4}\int d^2\theta \left(\frac{1}{g_c^2} - i\frac{\theta_{\mathrm{YM}}}{8\pi^2}\right) W^a(g_c V_c) W^a(g_c V_c) + h.c. \qquad (14.48)$$

These are not equivalent under a change of variables $V_h = g_c V_c$ in the Path Integral because there is a rescaling anomaly:

$$\begin{aligned}&\mathcal{D}(g_c V_c) = \qquad (14.49) \\ &\mathcal{D}V_c \exp\left(\frac{-i}{4}\int d^2\theta \left(\frac{2T(Ad)}{8\pi^2}\ln(g_c)\right) W^a(g_c V_c) W^a(g_c V_c) + h.c.\right)\end{aligned}$$

So

$$\begin{aligned}Z &= \mathcal{D}V_h \exp\left(\frac{i}{4}\int d^2\theta \frac{1}{g_h^2} W^a(V_h) W^a(V_h) + h.c.\right) \qquad (14.50) \\ &= \mathcal{D}V_c \exp\left(\frac{i}{4}\int d^2\theta \frac{1}{g_h^2} - \left(\frac{2T(Ad)}{8\pi^2}\ln(g_c)\right) W^a(g_c V_c) W^a(g_c V_c) + h.c.\right)\end{aligned}$$

So

$$\frac{1}{g_c^2} = \mathrm{Re}\left(\frac{1}{g_h^2}\right) - \frac{2T(Ad)}{8\pi^2}\ln(g_c) \qquad (14.51)$$

Adding matter fields Q_j, there is a similar effect from wavefunction renormalization:

$$Q'_j = Z_j(\mu, \mu')^{-1/2} Q_j \tag{14.52}$$

This rescaling anomaly is determined by the axial anomaly, which we calculated for $Z = e^{i\alpha}$:

$$\begin{aligned}\mathcal{D}(e^{i\alpha/2}Q'_j)\mathcal{D}(e^{-i\alpha/2}Q'^{\dagger}_j) &= \mathcal{D}Q'_j\mathcal{D}Q'^{\dagger}_j \qquad (14.53)\\ &\quad \exp\left(\frac{-i}{4}\int d^2\theta\left(\frac{T(r_j)}{8\pi^2}\ln(e^{i\alpha})\right)W^aW^a + h.c.\right)\end{aligned}$$

Thus

$$\frac{1}{g_c^2} = \mathrm{Re}\left(\frac{1}{g_h^2}\right) - \frac{2T(Ad)}{8\pi^2}\ln(g_c) - \sum_j \frac{T(r_j)}{8\pi^2}\ln(Z_j) \tag{14.54}$$

This leads precisely to the NSVZ β function. Since the relation between the two couplings is logarithmic, one cannot be expanded in a Taylor series around zero in the other.

References

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