AN EFFECTIVE FIELD THEORY FOR NON-RELATIVISTIC BOUND STATES IN QED AND QCD

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Luke, Rothstein, A.M., Phys. Rev. **D61** (2000) 074025

I. Stewart, A.M., Phys. Rev. **D62** (2000) 014033, hep-ph/0003032, 0003107, 0004018

J. Soto, I. Stewart, A.M., hep-ph/0006096

Earlier Work:

Caswell and Lepage (NRQED)

Bodwin, Braaten, Lepage (NRQCD)

Labelle

Luke, A.M.

Grinstein, Rothstein

Pineda, Soto (pNRQCD)

Brambilla, Vairo, Pineda, Soto

Yndurain, Titard

Beneke, Smirnov

MULTISCALE PROBLEM:

$$m \qquad p \sim mv \qquad E \sim mv^2 \qquad v \ll 1$$

For a Coulomb system: $v \sim \alpha$ (ignore Λ_{QCD})

GOAL:

1. Separate scales using an effective field theory
2. Sum large logarithms using the renormalization group

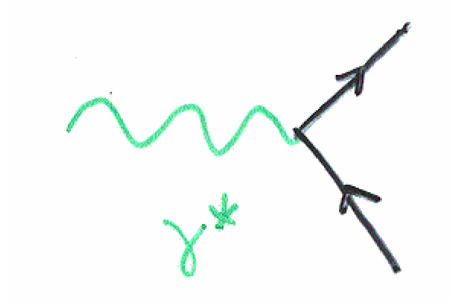
$$\ln \frac{p}{m}, \quad \frac{1}{2} \ln \frac{E}{m}, \quad \ln \frac{E}{p} \to \ln v \to \ln \alpha$$

3. Determine scale for α_s:

$$\alpha_s(m), \qquad \alpha_s(mv), \qquad \alpha_s(mv^2)$$

QCD

$\bar{t}t$ production near threshold; Υ



Large ratio of scales for $\bar{t}t$:

m	mv	mv^2	v
175 GeV	30 GeV	5 GeV	0.14

$\alpha_s \ln v$ not small, resumming logarithms is important

QED

$\alpha \ln \alpha$ small, BUT the experiments have high precision

H Lamb shift (2S–2P) (Lundeen, Pipkin)
 1057.845(9) MHz

H hyperfine splitting (Hellwig)
 1420.405 751 766 7(9) MHz

Muonium HFS (Mariam et al.)
 4 463.302 88(16) MHz

$$
\begin{aligned}
m_e \alpha^2 &\sim 4 \times 10^{10} \text{ MHz} \\
m_e \alpha^8 \ln^3 \alpha &\sim 743 \text{ KHz}
\end{aligned}
$$

In QED:

1. Find a universal description of $\ln\alpha$ terms.
   A single RG equation gives the Lamb shift, hyperfine splitting and decay widths for Hydrogen, Muonium, and o,p-Positronium
2. Understand the structure of the series and why they terminate
   - LO series: $\alpha^5 \ln\alpha$
   - NLO series: $\alpha^6 \ln\alpha$, $\alpha^7 \ln^2\alpha$, $\alpha^8 \ln^3\alpha$
3. some ∞ series: $\alpha^2 \ln\alpha \left(\alpha^3 \ln^2\alpha\right)^n$.

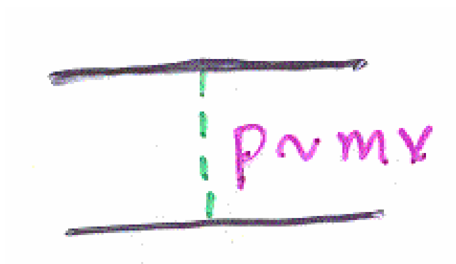
In QCD:

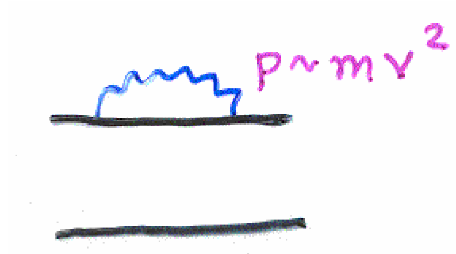
1. Sort out scale for α_s
2. RG improve the QCD potentials for bound states
3. ...

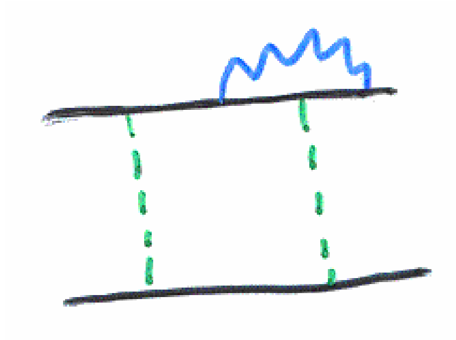
What is the problem?

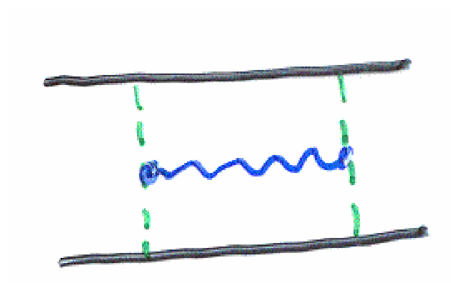
Hydrogen:

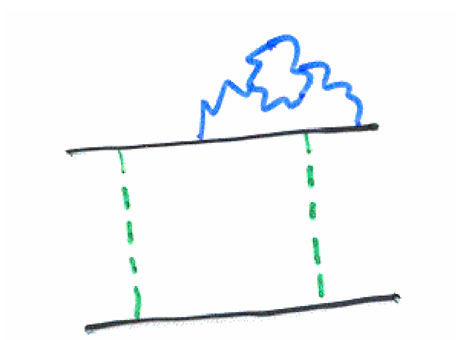
$$E_n = -\frac{m\alpha^2}{2n^2}, \qquad p \sim m\alpha, \qquad r \sim \frac{1}{m\alpha}, \qquad v \sim \alpha$$

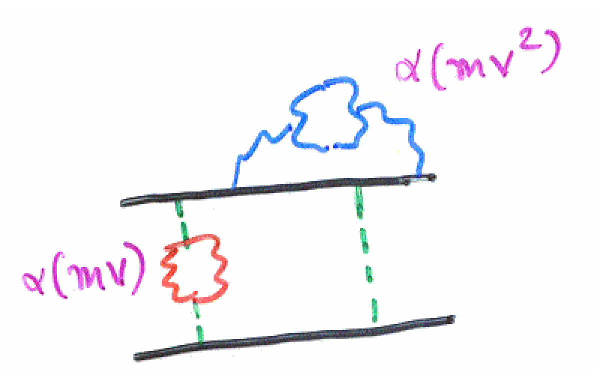












Degrees of Freedom

(Beneke, Smirnov)

Hard $\longleftarrow$

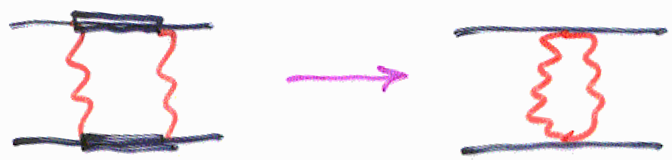
$E \sim p \sim m$

integrate these out

Soft $\longleftarrow$

$E \sim p \sim mv$

running of V_{Coulomb}



(Griesshammer)

Ultrasoft $\longleftarrow$

$E \sim p \sim mv^2$

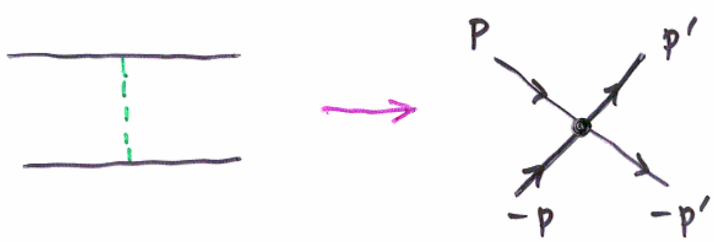
must multipole expand

(Labelle; Grinstein, Rothstein)

Potential

$E \sim mv^2$, $\leftarrow$

$p \sim mv$



(Pineda, Soto)

Effective Field Theory

Expansion in E/m, p/m does not work.

An expansion in v. (Count α as order v).

To have a consistent power counting, one finds (at scale m):

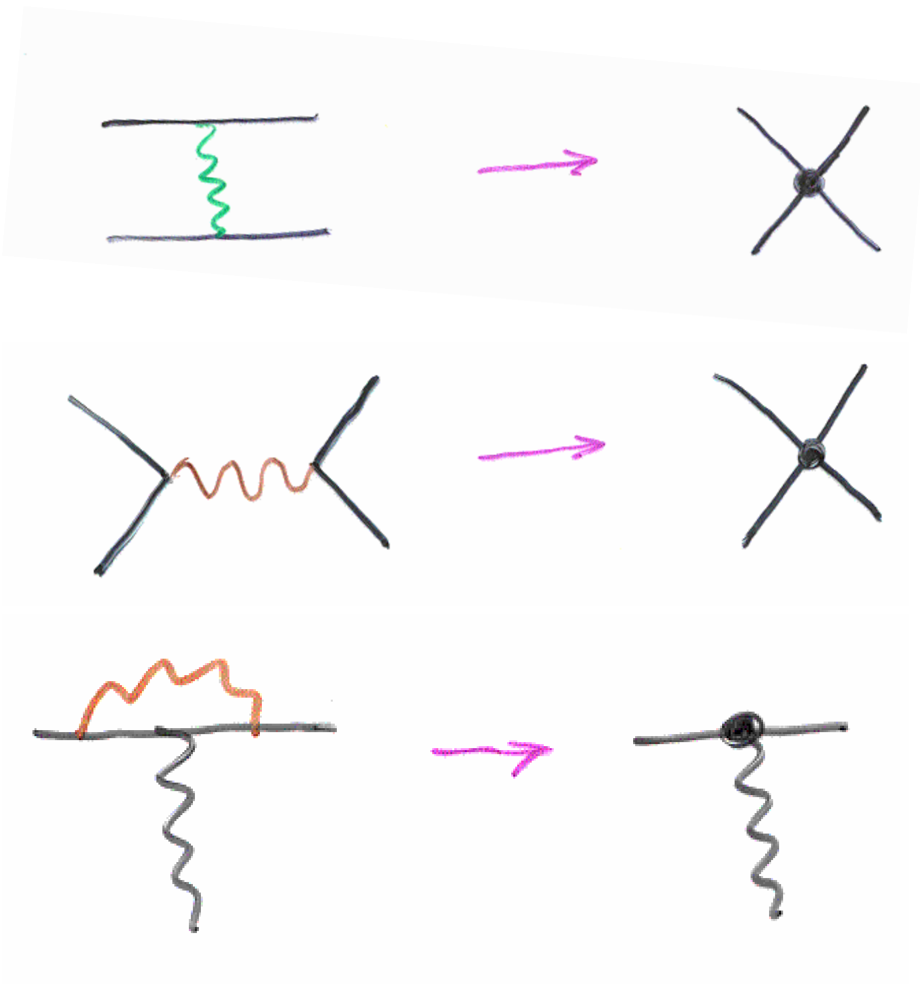
- Non-relativistic fermions with propagator

$$\frac{1}{E - \mathbf{p}^2/2m}$$

(fermions don't move in the static limit)

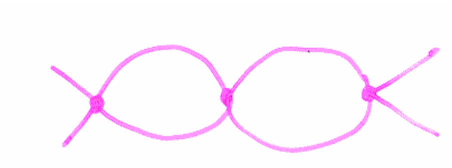
- Ultrasoft photons coupled via $\mathbf{p} - e\mathbf{A}$ interactions and multipole expanded (Labelle; Grinstein, Rothstein)
- Non-local potentials $V_{\text{eff}}(\mathbf{p}, \mathbf{p}')$
- Soft photons (Griesshammer)
- Static theory not the $v \to 0$ limit or $m \to \infty$ limit

MATCHING CONDITIONS

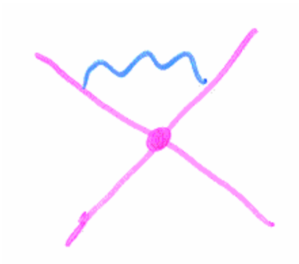


Velocity Renormalization Group

(Luke, Rothstein, A.M.; Stewart, A.M.)



$$\ln \frac{\sqrt{mE}}{\mu_S}$$



$$\ln \frac{E}{\mu_U}$$

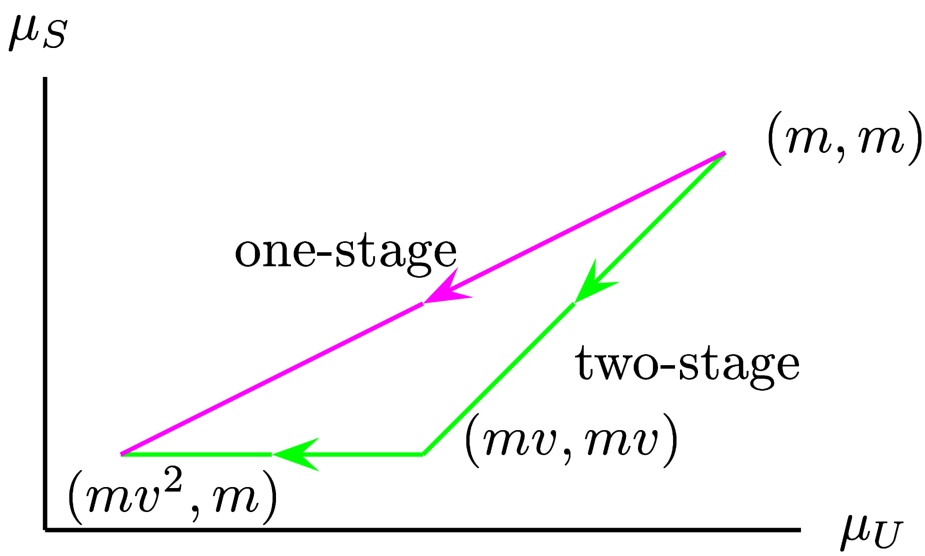
Two-stage running:

- Integrate μ_S, μ_U from $m \to mv$
- Integrate out soft modes
- Integrate μ_U from $mv \to mv^2$

One-stage running:

- Set $\mu_S = m\nu$, $\mu_U = m\nu^2$ and integrate from $\nu = 1$ to $\nu = v$.

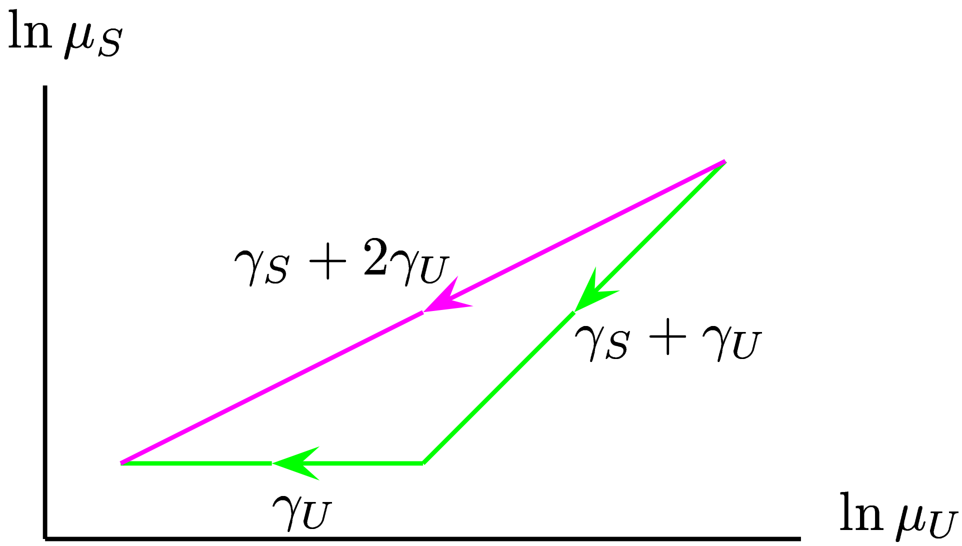
(Stewart, A.M.; Soto, Stewart, A.M.)



- Two methods give different answers, $\nabla \times \gamma \neq \mathbf{0}$.
- One-stage VRG agrees with explicit QED calculations at $\alpha^7 \ln^2 \alpha$ and $\alpha^8 \ln^3 \alpha$
- Generic result for correlated scales: mv and mv^2 not independent.

RUN IN VELOCITY, NOT MOMENTUM

Define $\gamma_S = \frac{d}{d\ln\mu_S}$, $\gamma_U = \frac{d}{d\ln\mu_U}$



Two-stage	One-stage
$\gamma_S + \gamma_U \quad m \to mv$ $\gamma_U \quad mv \to mv^2$	$\gamma_S + 2\gamma_U \quad 1 \to v$
$(\gamma_S + \gamma_U)\ln v + \gamma_U \ln v$	$(\gamma_S + 2\gamma_U)\ln v$

Single-log terms agree. BUT γ not constants, and depend on couplings, $\gamma(V)$, and V can run.

So one gets:

$$\gamma_S\,(\gamma_U)\ln^2 v \qquad \gamma_S(2\gamma_U)\ln^2 v$$
$$\gamma_S\,(\gamma_U)^2\ln^3 v \qquad \gamma_S(2\gamma_U)^2\ln^3 v$$

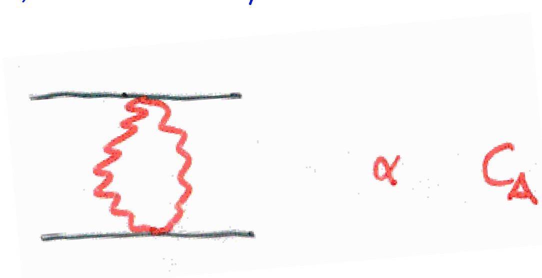
RUNNING POTENTIAL

$$V(\mathbf{p},\mathbf{p}') = V^{(-1)} + V^{(0)} + V^{(1)} + V^{(2)} + \ldots$$

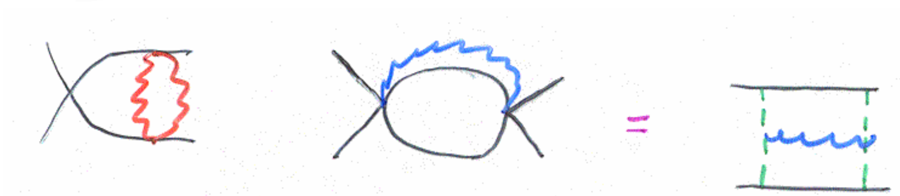
$$V^{(-1)} = \frac{U_c}{\mathbf{k}^2}\,,$$
$$V^{(0)} = \frac{U_k}{|\mathbf{k}|}\,,$$
$$V^{(1)} = U_2 + U_s\,\mathbf{S^2} + \frac{U_r(\mathbf{p}^2+\mathbf{p}'^2)}{2\mathbf{k}^2} - \frac{i\mathbf{U}_\Lambda\cdot(\mathbf{p}'\times\mathbf{p})}{\mathbf{k}^2}$$
$$+U_t\left(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \frac{3\,\mathbf{k}\cdot\boldsymbol{\sigma}_1\,\mathbf{k}\cdot\boldsymbol{\sigma}_2}{\mathbf{k}^2}\right),$$

$V^{(0)} \sim 1/m, V^{(1)} \sim 1/m^2$

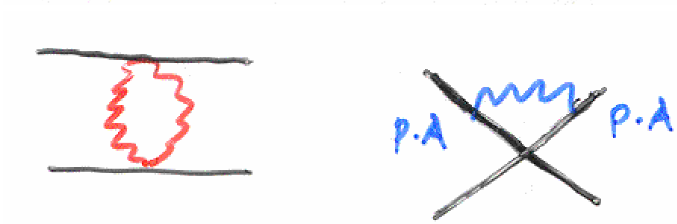
$V^{(-1)}$



$V^{(0)}$



$V^{(1)}$



Potentials for $t\bar{t}$

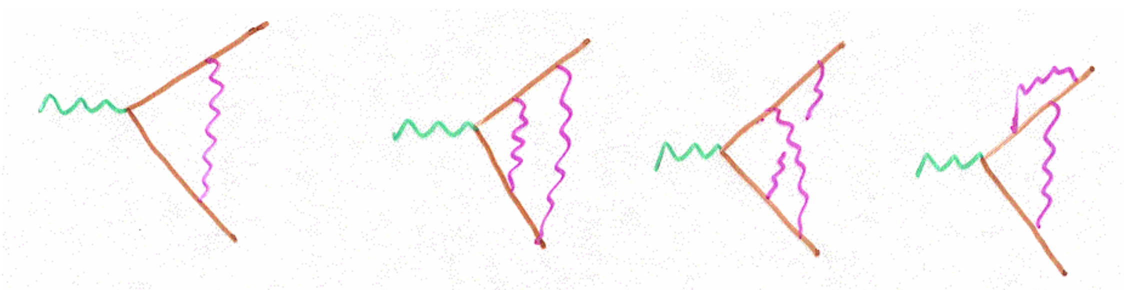
(Stewart, A.M.)

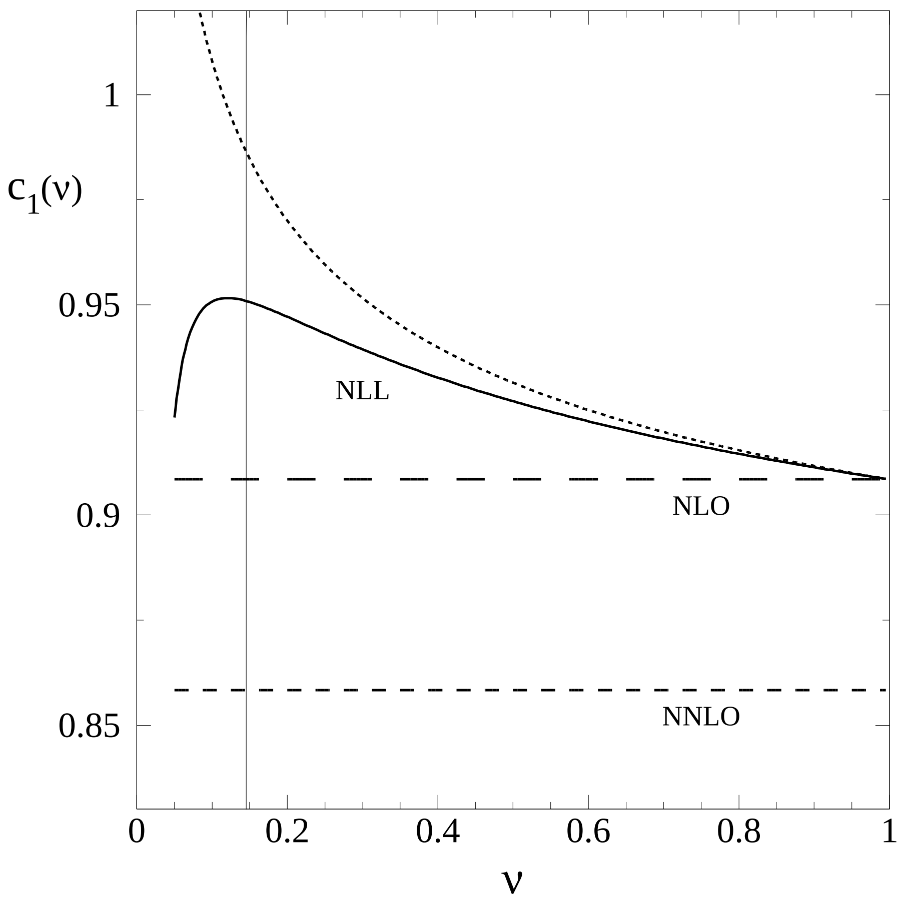
	$\mathcal{V}_r^{(s)}$	$\mathcal{V}_2^{(s)}$	$\mathcal{V}_s^{(s)}$	$\mathcal{V}_\Lambda^{(s)}$	$\mathcal{V}_t^{(s)}$
$\nu = 1$	-1.81	0	0.60	0.15	2.71
$\nu = v$	-1.49	0.63	0.53	0.16	3.11

$$\begin{aligned}\mathcal{V}_2^{(1)}(\nu) &= \frac{14C_1}{3}\alpha_s(m\nu)\alpha_s(m)\ln\left(\frac{m\nu}{m}\right)\\ &\quad -\frac{32\pi C_1}{3\beta_0}\alpha_s(m)\ln\left[\frac{\alpha_s(m\nu)}{\alpha_s(m\nu^2)}\right]\end{aligned}$$

$$c_1 = 1 - \frac{2C_F\alpha_s}{\pi} + \alpha_s^2 C_F\left(\frac{1}{3}C_F + \frac{1}{2}C_A\right)\ln\frac{m}{\mu} + \text{non-log}$$

(Czarnecki, Melnikov; Beneke, Signer, Smirnov)




c1(ν)
1
0.95
0.9
0.85
0
0.2
0.4
0.6
0.8
1
ν
NLL
NLO
NNLO

QED

			order	E
$V^{(-1)}$	$\frac{\alpha}{\mathbf{k}^2}$	$\frac{\alpha}{v}$	1	α^2
$V^{(0)}$	$\frac{\alpha^2}{m\lvert\mathbf{k}\rvert}$	α^2	α^2	α^4
$V^{(1)}$	$\frac{\alpha}{m^2},\ \frac{\alpha\mathbf{S}^2}{m^2}$	αv	α^2	α^4
$V^{(2)}$	$\frac{\alpha^2\lvert\mathbf{k}\rvert}{m^3}$	$\alpha^2 v^2$	α^4	α^6
$V^{(3)}$	$\frac{\alpha\mathbf{k}^2}{m^4}$	αv^3	α^4	α^6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$T\left(V^{(a)}V^{(b)}\right) \sim V^{(a+b)}$$

1. Sum Coulomb potential to all orders
2. $$V^{(0)}V^{(0)} \sim \alpha^4 v^0 \to \alpha^6 \text{ in E}$$
   $$V^{(0)}V^{(1)} \sim \alpha^3 v \to \alpha^6 \text{ in E}$$
   $$V^{(1)}V^{(1)} \sim \alpha^2 v^2 \to \alpha^6 \text{ in E}$$
3. To order α^4 in energy, need only
   $$\left\langle V^{(0)} + V^{(1)} \right\rangle$$

Define LO and NLO anomalous dimensions relative to the leading term.

$\gamma_{\rm LO}, \gamma_{\rm NLO}$ are order α^2, α^3 for $V^{(1)}$

$\gamma_{\rm LO}, \gamma_{\rm NLO}$ are order α^3, α^4 for $V^{(0)}$

Since V's are of different orders, LO/NLO not related to the number of loops.

$$
\begin{aligned}
\gamma_{\rm LO} &: \quad \alpha^4 \quad\;\; \left(1+\alpha\ln\alpha+\alpha^2\ln^2\alpha+\alpha^3\ln^3\alpha+\ldots\right)\\
\gamma_{\rm NLO} &: \quad \alpha^4\alpha \;\; \left(1+\alpha\ln\alpha+\alpha^2\ln^2\alpha+\alpha^3\ln^3\alpha+\ldots\right)\\
\gamma_{\rm NNLO} &: \quad \alpha^4\alpha^2 \left(1+\alpha\ln\alpha+\alpha^2\ln^2\alpha+\alpha^3\ln^3\alpha+\ldots\right)
\end{aligned}
$$

$\gamma_{\rm NNLO}$ the same order as $\gamma_{\rm LO}$ for $V^{(2,3)}$.

So one can compute

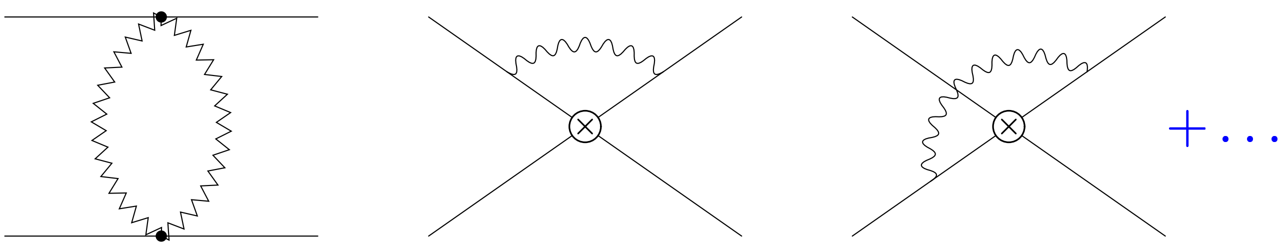
$$
\begin{array}{cccc}
\alpha^5\ln\alpha & \alpha^6\ln^2\alpha & \alpha^7\ln^3\alpha & \ldots\\
\alpha^6\ln\alpha & \alpha^7\ln^2\alpha & \alpha^8\ln^3\alpha & \ldots
\end{array}
$$

using $\gamma_{\rm LO}$, $\gamma_{\rm NLO}$ for $V^{(0,1)}$.

$$V^{(-1)} = \frac{U_c}{\mathbf{k}^2} \qquad V^{(0)} = \frac{U_k}{|\mathbf{k}|},$$

$$V^{(1)} = U_2 + U_s\,\mathbf{S^2} + \frac{U_r(\mathbf{p}^2 + \mathbf{p}'^2)}{2\mathbf{k}^2} - \frac{i\mathbf{U}_\Lambda \cdot (\mathbf{p}' \times \mathbf{p})}{\mathbf{k}^2}$$
$$+U_t\left(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - \frac{3\,\mathbf{k}\cdot\boldsymbol{\sigma}_1\,\mathbf{k}\cdot\boldsymbol{\sigma}_2}{\mathbf{k}^2}\right),$$

Coulomb potential does not run in QED



(a) (b) (c)

At LO,

$$\nu\frac{dU_k}{d\nu} = 0 \qquad \propto C_A \text{ in QCD}$$

$$\nu\frac{dU_2}{d\nu} = \frac{2\alpha}{3\pi}\left(\frac{1}{m_1} + \frac{Z}{m_2}\right)^2 U_c + \frac{14Z^2\alpha^2}{3m_1m_2} = \gamma_0 U_c$$

$$\nu\frac{dU_3}{d\nu} = \frac{2\alpha}{3\pi}\left(\frac{1}{m_1} + \frac{Z}{m_2}\right)^2 U_k + \gamma_1 U_c + \gamma_2 U_c^2$$

Particles $(-e, m_1)$ and (Ze, m_2)

$$\gamma_0 = \frac{2\alpha}{3\pi}\left(\frac{1}{m_1^2} + \frac{Z}{4m_1 m_2} + \frac{Z^2}{m_2^2}\right)$$

γ_0 is a constant in QED, since α does not run

Integrate:

$$U_2(\nu) = U_2(1) + \gamma_0 U_c \ln \nu$$

Only a single term, so the LO series terminates

No terms in $\alpha^4 (\alpha \ln \alpha)^n$ series except for $n = 1$.
Power counting guarantees nothing left out at arbitrarily high orders.

$$\begin{aligned}
\Delta E &= \langle U_2 \rangle \\
&= \gamma_0 U_c \ln \nu \, |\psi(0)|^2 \\
&= -\frac{8Z^4\alpha^5 m_R^3}{3\pi n^3}\left(\frac{1}{m_1^2} + \frac{Z}{4m_1 m_2} + \frac{Z^2}{m_2^2}\right) \ln Z\alpha,
\end{aligned}$$

H—(Bethe 1947)

$$|\psi(0)|^2 = \frac{(m_R Z\alpha)^3}{\pi n^3} \qquad n\text{S state}$$

NLO

$$\nu \frac{dU_k}{d\nu} = \gamma_3 U_c$$

$$\nu \frac{dU_{2+s}}{d\nu}\bigg|_{\rm NLO} = \rho_{ccc}\, U_c^3 + \rho_{cc2}\, U_c^2 \left(U_{2+s} + U_r\right)$$
$$+\rho_{c22}\, U_c \left(U_{2+s}^2 + 2U_{2+s}U_r + \frac{3}{4}U_r^2 - 9U_t^2 \mathbf{S^2}\right)$$
$$+\rho_{ck}\, U_c U_k + \rho_{k2}\, U_k \left(U_{2+s} + U_r/2\right)$$
$$+\rho_{c3}\, U_c \left(U_3 + U_{3s} S^2 + \frac{1}{2} U_{rk}\right) + \rho_s \frac{Z^3\alpha^3}{m_1 m_2},$$

where $U_{2+s} = U_2 + U_s \mathbf{S}^2$ and $\rho_{c22} = -m_R^2/4\pi^2$.

Can integrate RHS using LO values for U_i.

Only terms which run at LO are U_2 and U_3.

$$\int \text{const} = \ln \nu$$
$$\int \ln \nu = \frac{1}{2} \ln^2 \nu$$
$$\int \ln^2 \nu = \frac{1}{3} \ln^3 \nu$$

NLO series terminates after 3 terms

$$\alpha^6 \ln \alpha, \qquad \alpha^7 \ln^2 \alpha, \qquad \alpha^8 \ln^3 \alpha$$

Anomalous Dimensions

$$
\begin{aligned}
\rho_{ccc} &= -\frac{m_R^4}{64\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)^2, \qquad \rho_{c22} = -\frac{m_R^2}{4\pi^2},\\
\rho_{cc2} &= -\frac{m_R^3}{8\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right), \qquad \rho_{c3} = \frac{2m_R}{\pi^2},\\
\rho_{ck} &= \frac{m_R^2}{2\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right), \qquad \rho_{k2} = \frac{2m_R}{\pi^2}.
\end{aligned}
$$

$\ln^3 \alpha$

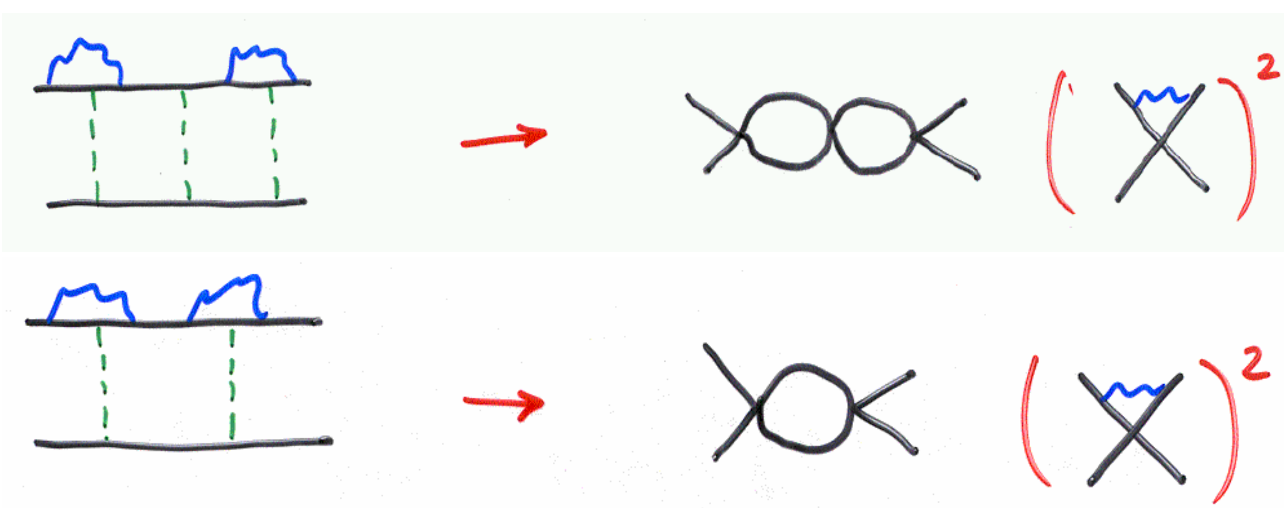
$$\frac{1}{3}\,\gamma_0^2\,\rho_{c22}\,U_c^3(1)\,\ln^3\nu,$$

Lamb shift for the nS state (no HFS, Γ)

$$\begin{aligned}\Delta E &= \frac{64 m_R^5 \alpha^8 Z^6}{27\pi^2 n^3}\left(\frac{1}{m_1^2}+\frac{Z}{4m_1m_2}+\frac{Z^2}{m_2^2}\right)^2 \ln^3(Z\alpha)\\ &= \frac{3m_e\alpha^8\ln^3\alpha}{8\pi^2 n^3} \quad \text{(positronium)}\end{aligned}$$

(8 KHz for Hydrogen 2P–2S)

Karshenboim 1993	$a_{63} = -8/27$
Malampalli and Sapirstein PRL 1998	$a_{63} = -0.652$
Goidenko et al. PRL 1999	$a_{63} = -0.296$
Yerokhin hep-ph/0001327	$a_{63} = -0.652$



$\ln^2 \alpha$

$$\gamma_0\, \rho_{c22}\, U_c^2(1) \left[U_2(1) + U_s(1)\mathbf{S}^2\right] \ln^2 \nu + \ldots$$

$$\text{HFS:} -\frac{64 Z^6 \alpha^7 m_R^5 \mu_1 \mu_2}{9 m_1 m_2 \pi n^3} \left[\frac{1}{m_1^2} + \frac{Z}{4 m_1 m_2} + \frac{Z^2}{m_2^2}\right] \ln^2 (Z\alpha),$$

(Karshenboim 93, Labelle 94)

$$\text{Ps HFS} \qquad -\frac{7 m_e}{8\pi n^3}\, \alpha^7 \ln^2 \alpha,$$

(Melnikov and Yelkhovsky 99) $\surd$

$$\frac{\Delta\Gamma}{\Gamma_0} = \gamma_0\, \rho_{c22}\, U_c(1)^2 \ln^2 \nu = -\frac{3}{2\pi} \alpha^3 \ln^2 \alpha,$$

(Karshenboim 93) $\surd$

Lamb Shift needs γ_1, γ_2

$\ln \alpha$

$$U_{2+s}\left[\rho_{c22}U_c\left(U_{2+s}+2U_r\right)+\rho_{cc2}U_c^2+\rho_{2k}U_k\right]\ln\nu+\ldots$$

$$\frac{\Delta\Gamma}{\Gamma_0}=\left(\frac{m_e^2}{2\pi}\mathrm{Re}\,U_{2+s}-2\right)\ln\nu=\left(\frac{7\mathbf{S}^2}{6}-2\right)\alpha^2\ln\alpha,$$

$$\left(\frac{\Delta\Gamma}{\Gamma_0}\right)_{\rm ortho}=\frac{\alpha^2}{3}\ln\alpha\,,\qquad\left(\frac{\Delta\Gamma}{\Gamma_0}\right)_{\rm para}=-2\alpha^2\ln\alpha\,,$$

(Caswell and Lepage 79, Khriplovich and Yelkhovsky 90) $\surd$

HFS and Lamb Shift depend on γ_3 and ρ_s

Infinite Series

$$V(\nu = Z\alpha) = \exp\left[\frac{2\alpha}{3\pi}\left(\frac{1}{m_1}+\frac{Z}{m_2}\right)^2 \mathbf{k}^2 \ln Z\alpha\right]\frac{U_c(1)}{\mathbf{k}^2},$$

$$V(Z\alpha) = -\frac{Z\alpha}{r}\mathrm{Erf}\left[r\sqrt{\frac{3\pi}{8\alpha\ln[1/(Z\alpha)]}}\left(\frac{1}{m_1}+\frac{Z}{m_2}\right)^{-1}\right],$$

$\alpha^5 \ln Z\alpha(\alpha^3 \ln Z\alpha)^{n/2}$, $n \geq 0$.

$$\nu\frac{dU_2}{d\nu} = \gamma_0 U_c + \rho_{c22} U_c U_2^2.$$

$$U_2(\nu) = \frac{U_2(1) + \sqrt{\gamma_0/|\rho_{c22}|}\tanh\left[\sqrt{\gamma_0|\rho_{c22}|}\,U_c(1)\ln\nu\right]}{1+\sqrt{|\rho_{c22}|/\gamma_0}U_2(1)\tanh\left[\sqrt{\gamma_0|\rho_{c22}|}\,U_c(1)\ln\nu\right]},$$

$\alpha^2 \ln\alpha(\alpha^3 \ln^2 \alpha)^n$.

Conclusions

1. Systematic way to separate scales in non-relativistic bound states
2. All large logs summed using the velocity RG
3. Universal description of QED logs.
   Checks: $\alpha^5 \ln \alpha$, $\alpha^6 \ln \alpha$, $\alpha^7 \ln^2 \alpha$, $\alpha^8 \ln^3 \alpha$
4. QCD: can distinguish $\alpha(mv)$ and $\alpha(mv^2)$.
   Both can appear in the same equation
5. VRG: apply to other problems with correlated scales