The classical limit and entanglement for a class of quantum baker's maps

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based on:

Tracy M M and Scott A J *The classical limit for a class of quantum baker's maps* J. Phys. A **35** (2002) 8341.

Scott A J and Caves C M *Entangling power of the quantum baker's map* J. Phys. A (to be published) quant-ph/0305046.

Program

- Classical baker's map
- A class of quantum baker's maps
- Classical limit?
- Entangling power

Motivation

- Interest in the classical baker's map stems from its simplicity
  - ▷ displays all essential features of classical chaos
  - ▷ straightforward characterization in terms of symbolic dynamics
- A quantum baker's map would be an ideal candidate for the investigation of quantum chaos, however,
  - ▷ there is no unique quantization procedure
  - ▷ A complete understanding of quantum chaos requires the investigation of 'nonstandard' quantizations

Classical baker's map

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q_{n+1} &= 2q_n - \lfloor 2q_n \rfloor \\
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where

- $q, p \in [0,1)$
- $\lfloor x \rfloor$ is the integer part of x

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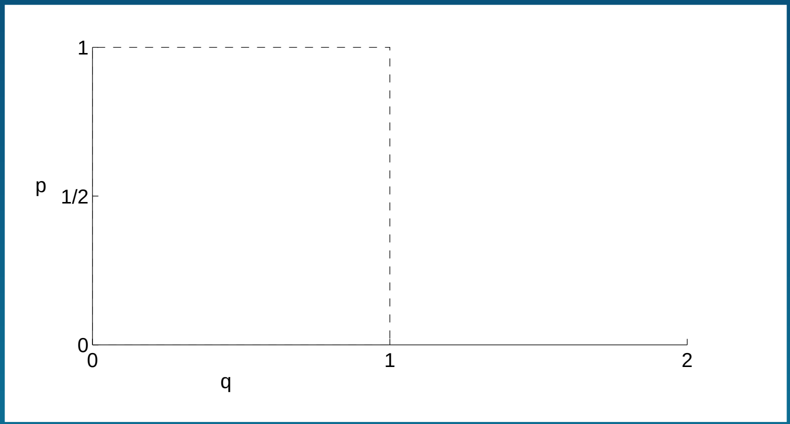

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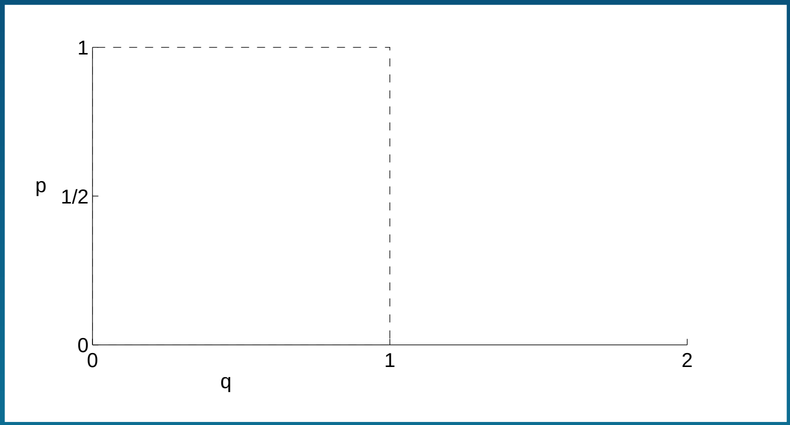

$(q, p) \quad \leftrightarrow \quad s = \ldots s_{-2}s_{-1}s_0 \bullet s_1s_2s_3\ldots$ (bi-infinite symbolic sequence)

where

- $q = 0 \cdot s_1s_2\ldots = \sum_{k=1}^{\infty} s_k 2^{-k}$
- $p = 0 \cdot s_0s_{-1}\ldots = \sum_{k=0}^{\infty} s_{-k} 2^{-k-1}$
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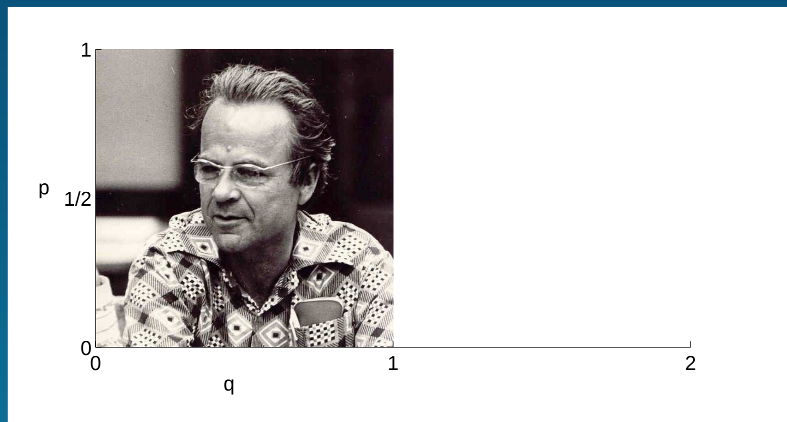

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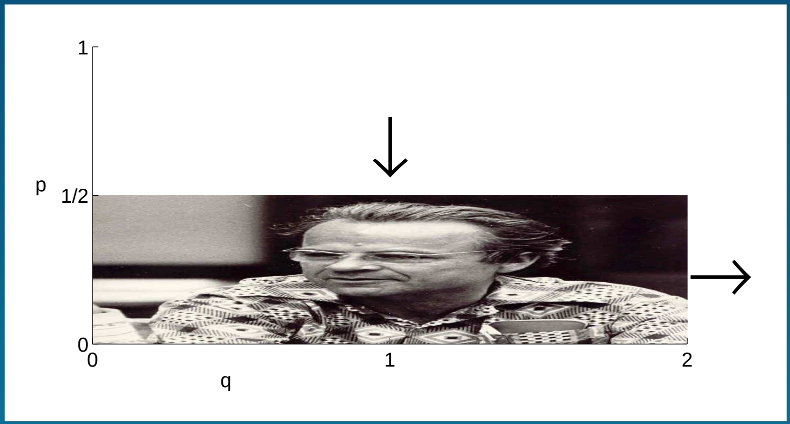

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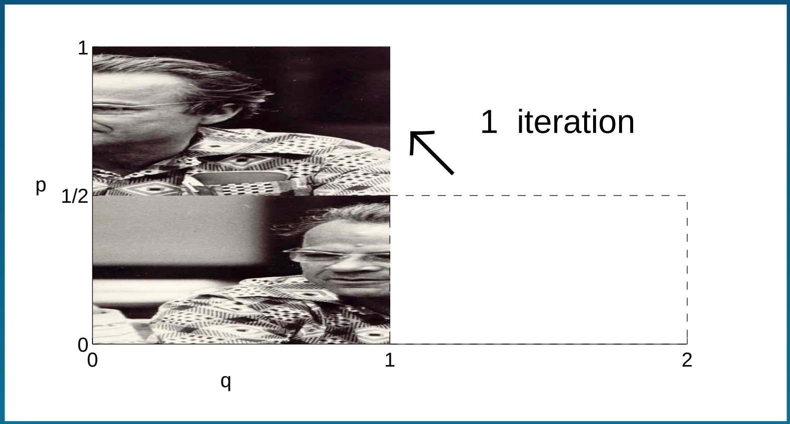

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Finite-dimensional Hilbert space

- Work in the D-dimensional Hilbert space $\mathcal{H}_D$
- spanned by either the position states $|q_j\rangle$, with eigenvalues $q_j = (j+1/2)/D$,

  or momentum states $|p_k\rangle$, with eigenvalues $p_k = (k+1/2)/D$,

  where $\langle q_j|q_k\rangle = \langle p_j|p_k\rangle = \delta_{jk} \qquad j,k = 0\ldots D-1$
- which are related via the finite Fourier transform

$$\langle q_j|\hat{F}|q_k\rangle \equiv \langle q_j|p_k\rangle = \frac{1}{\sqrt{D}}e^{iq_jp_k/\hbar}$$

- Both position and momentum are chosen to be antiperiodic:

$$|q_{j+D}\rangle = -|q_j\rangle \qquad |p_{k+D}\rangle = -|p_k\rangle \qquad \text{(toroidal phase space)}$$

- For consistency of units: $2\pi\hbar D \equiv 1$

Finite-dimensional Hilbert space

- Also define coherent states for $\mathcal{H}_D$:

$$|a\rangle \equiv \frac{1}{\mathcal{N}}\left(\frac{2}{D}\right)^{1/4} \sum_{j=0}^{D-1} \exp\left[-\frac{\pi D}{2}\left(|a|^2+a^2\right) - \pi D\left(q_j^2 - 2q_j a\right)\right] \cdot \theta_0\left[iD(q_j-a)\middle|iD\right]|q_j\rangle$$

where $a \equiv q+ip$, $|a \pm 1\rangle = -\exp[\pm\pi i Dp]|a\rangle$, $|a \pm i\,\rangle = -\exp[\mp\pi i Dq]|a\rangle$,

and θ_0 is called a theta function:

$$\theta_0\left[z\middle|\tau\right] \equiv \sum_{\mu=-\infty}^{\infty} \exp\left[i\pi\left(\tau\mu^2 + (2z+1)\mu\right)\right]$$

- The Husimi function is then $|\langle\psi|a\rangle|^2$

Quantum baker's map

- For dimensions $D = 2^N$
- model our space as the tensor product of N qubits with a binary expansion association

$$|q_j\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_N\rangle \qquad\qquad x_l \in \{0, 1\}$$

where $j \equiv x_1 \ldots x_N \cdot 0 = \sum_{l=1}^{N} x_l 2^{N-l}$ and $q_j \equiv \dfrac{j + 1/2}{D}$

Quantum baker's map

- Now define the partial Fourier transformed states

$$|a_{N-n}\dots a_1 \bullet x_1 \dots x_n\rangle \quad \equiv \quad \hat{G}_n \, |x_1\rangle \otimes \cdots \otimes |x_n\rangle \otimes |a_1\rangle \otimes \cdots \otimes |a_{N-n}\rangle$$

$$\equiv \quad |x_1\rangle \otimes \cdots \otimes |x_n\rangle \otimes \frac{1}{\sqrt{2^{N-n}}} \sum_{x_{n+1},\dots,x_N} |x_{n+1}\rangle \otimes \cdots \otimes |x_N\rangle e^{2\pi i a x/2^{N-n}}$$

where $\quad a \equiv a_1 \dots a_{N-n} \cdot 1 \quad$ and $\quad x \equiv x_{n+1} \dots x_N \cdot 1$

- The operator $\hat{G}_n$ is the partial Fourier transform
- $\hat{G}_0 = \hat{F} \quad$ and $\quad \hat{G}_N = i\hat{1}$

Quantum baker's map

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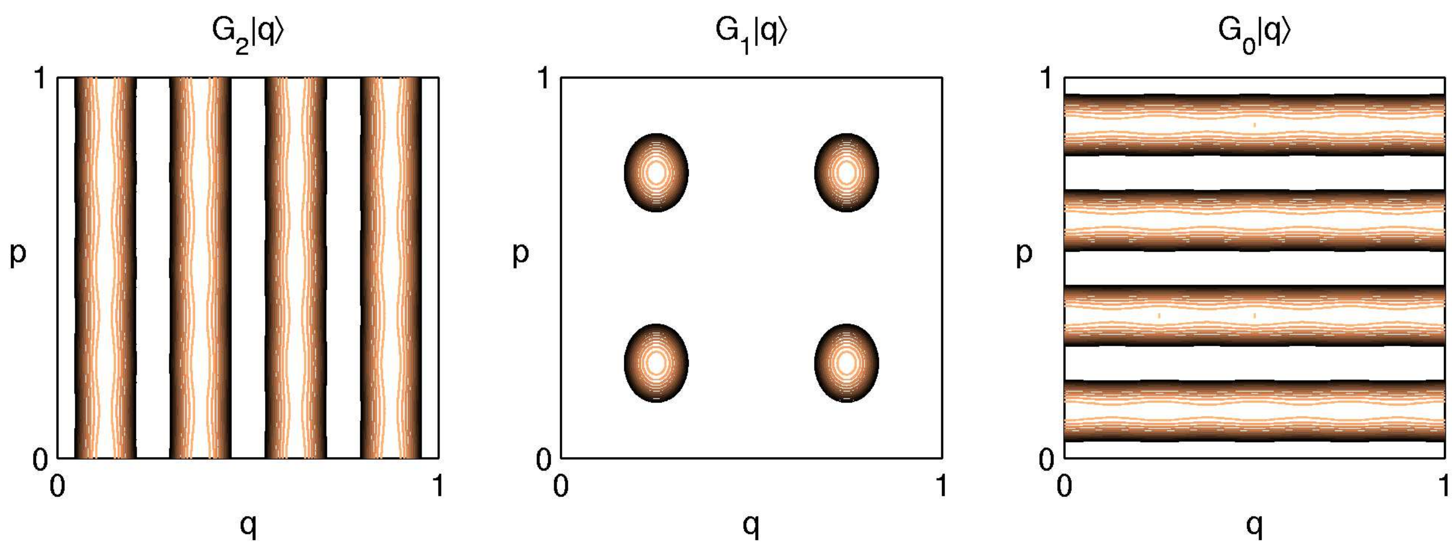

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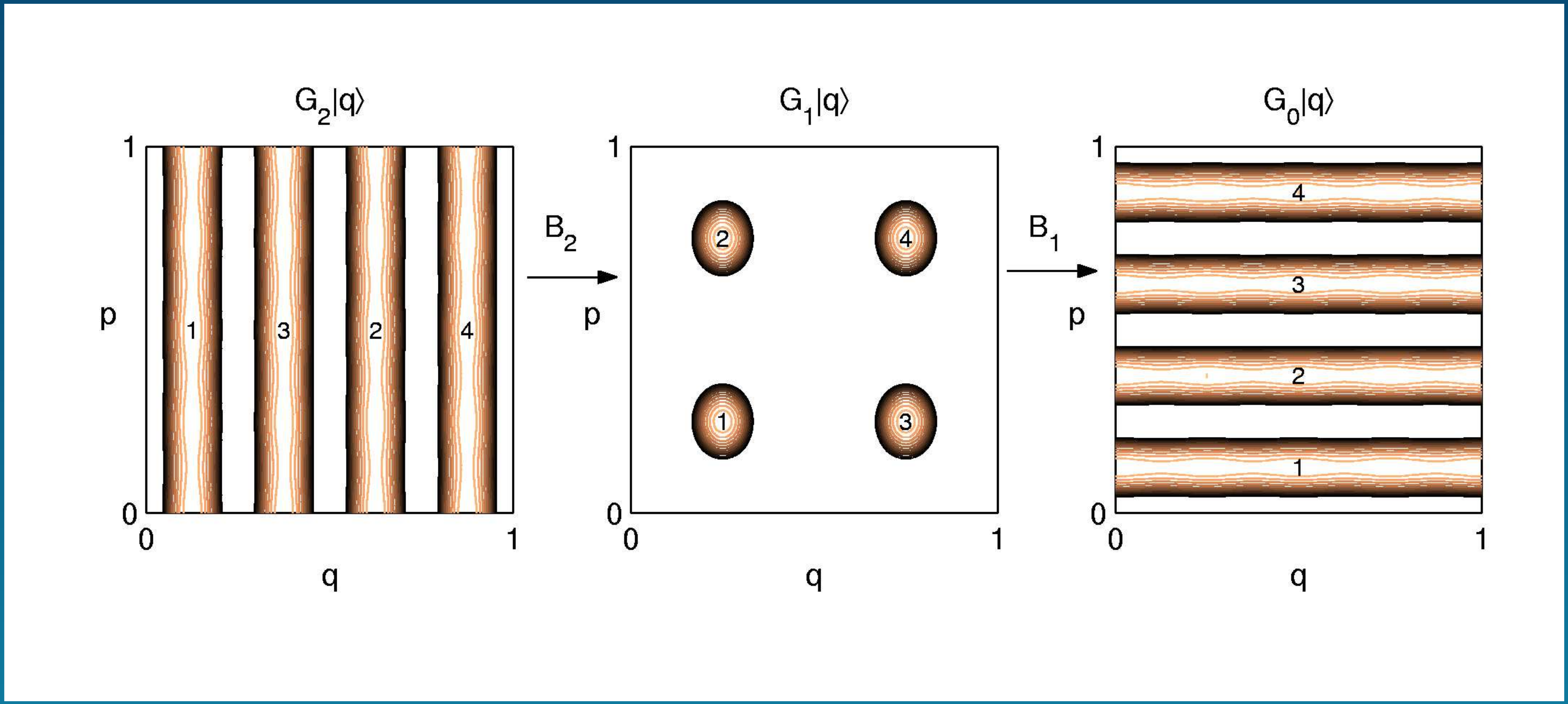

- Two different quantum baker's maps can be defined.

Quantum baker's map

- A whole class of quantum baker's maps $\hat{B}_n \quad (n = 1, \ldots, N)$ can be defined

$$\hat{B}_n = \sum_{\substack{x_1,\ldots,x_n \\ a_1,\ldots,a_{N-n}}} |a_{N-n} \ldots a_1 x_1 \bullet x_2 \ldots x_n\rangle\langle a_{N-n} \ldots a_1 \bullet x_1 x_2 \ldots x_n|$$

- This class of quantum baker's maps was introduced by Schack and Caves
- The original Balazs-Voros-Saraceno baker's map is the special case $n = 1$

Quantum baker's map

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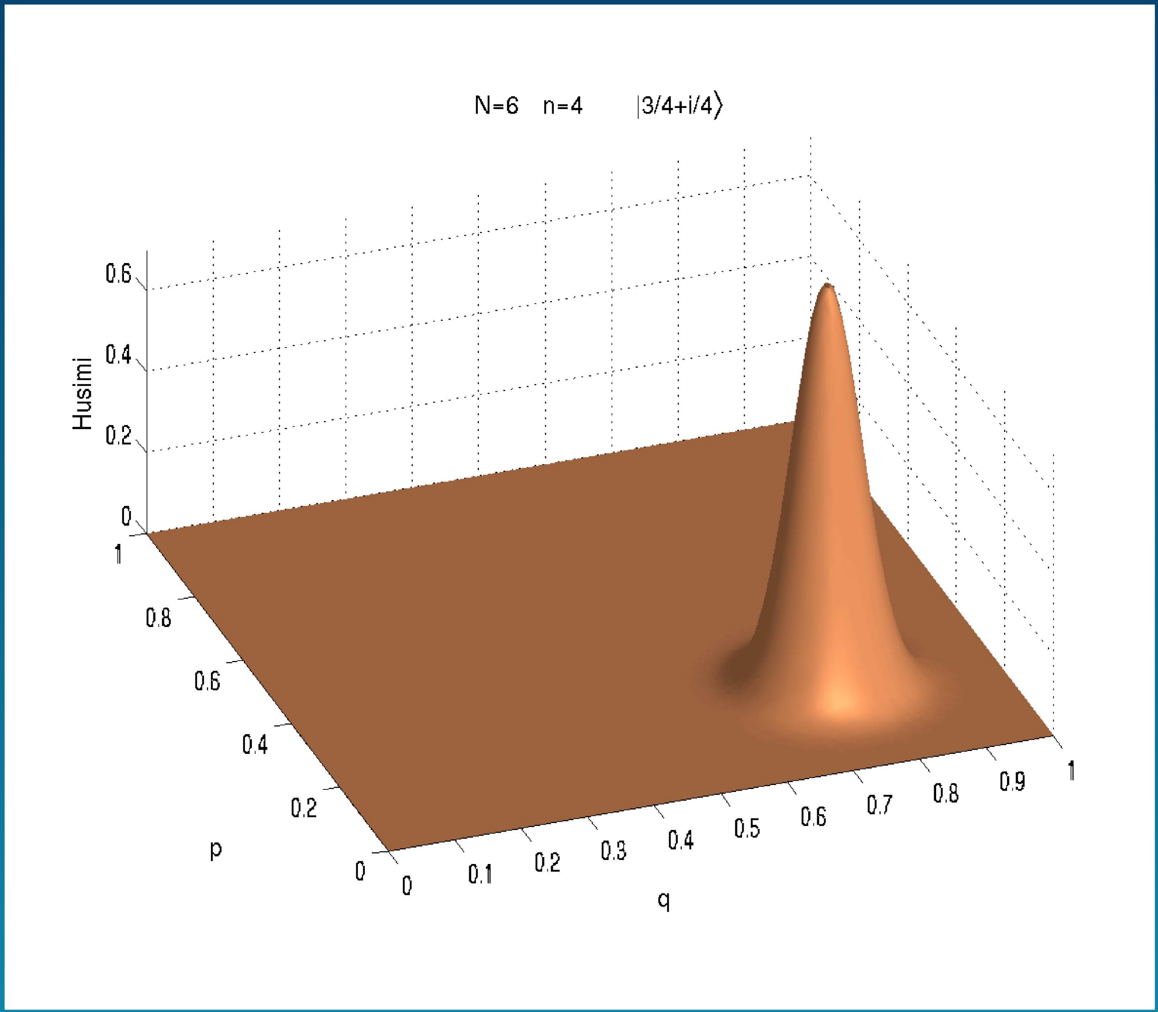

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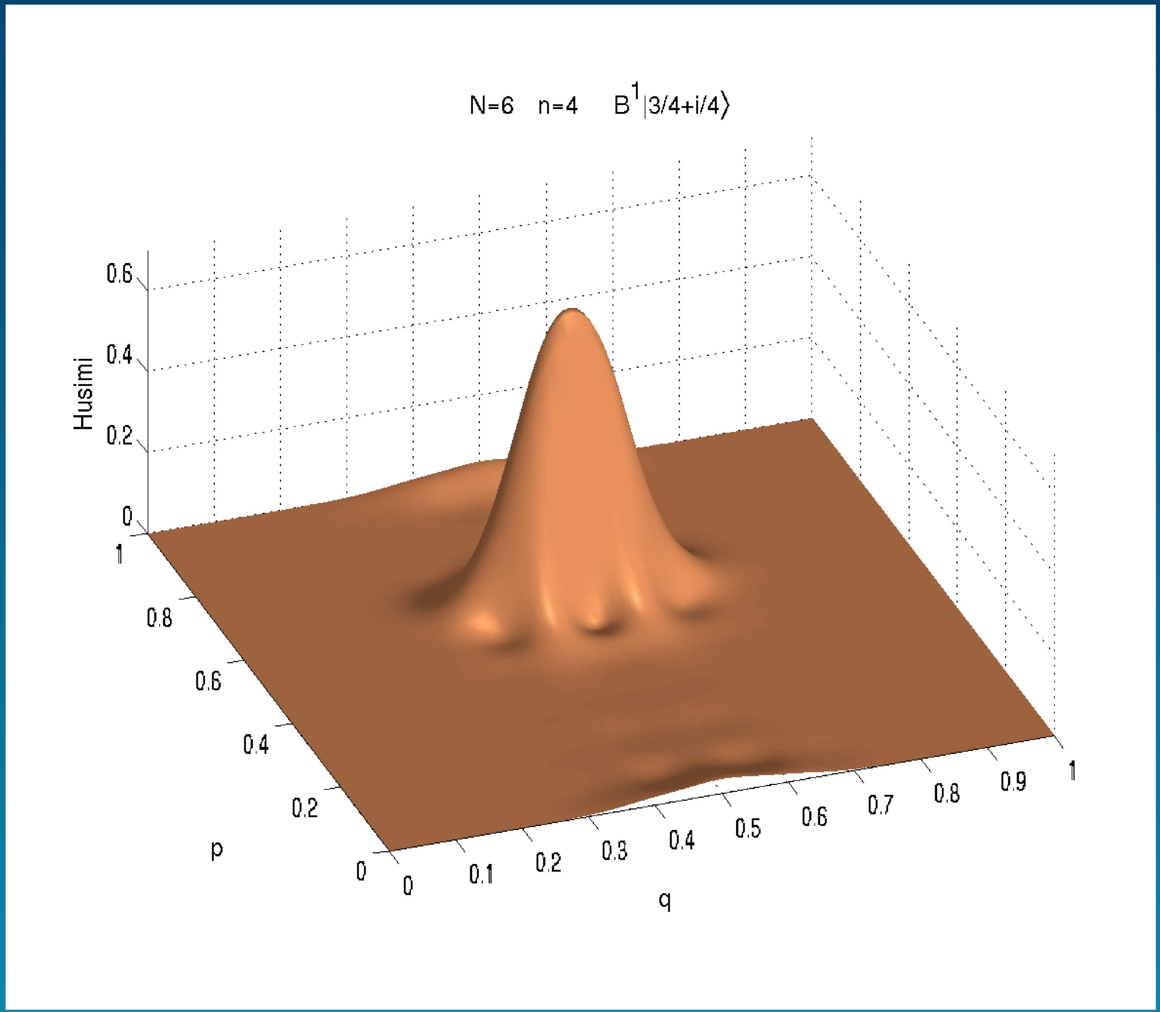

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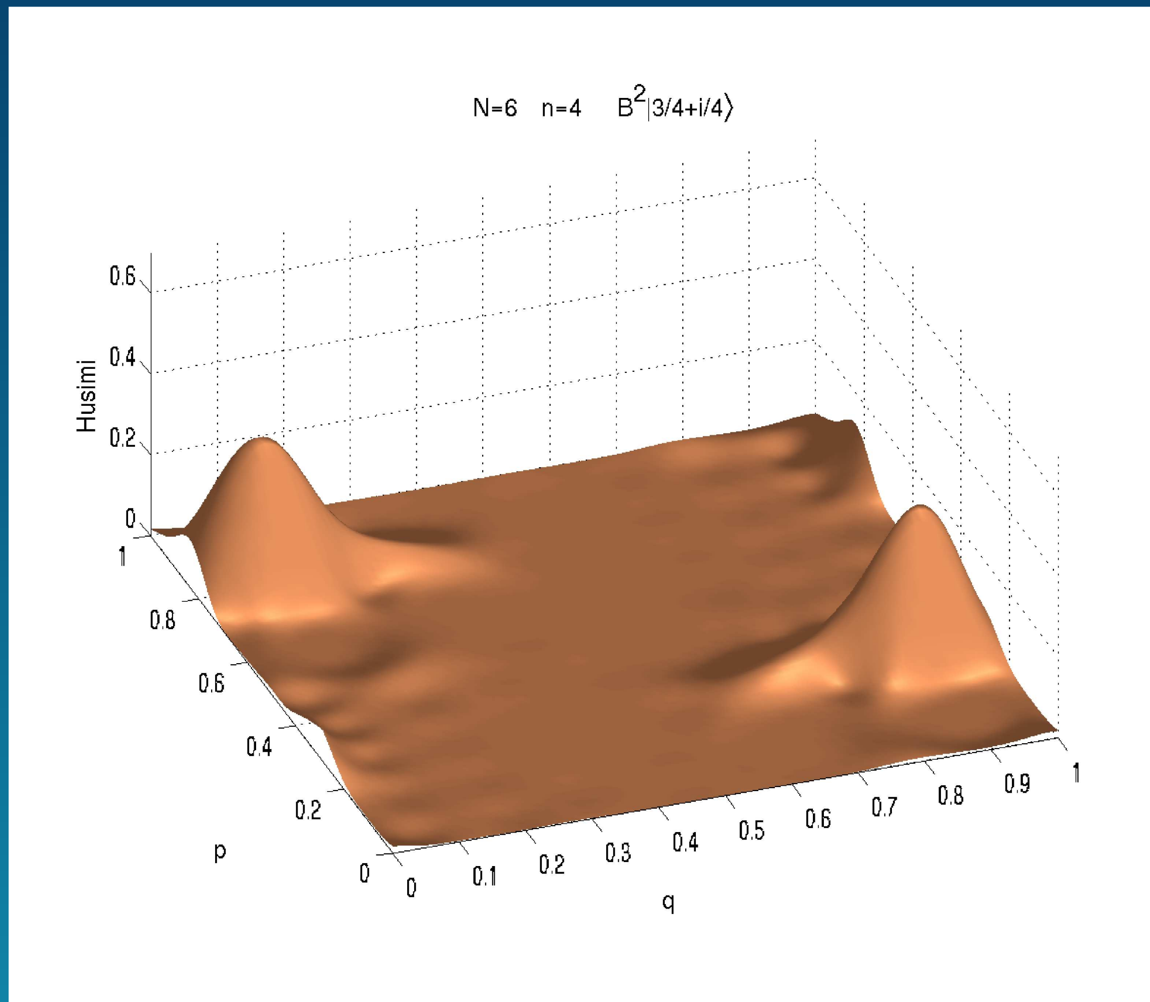

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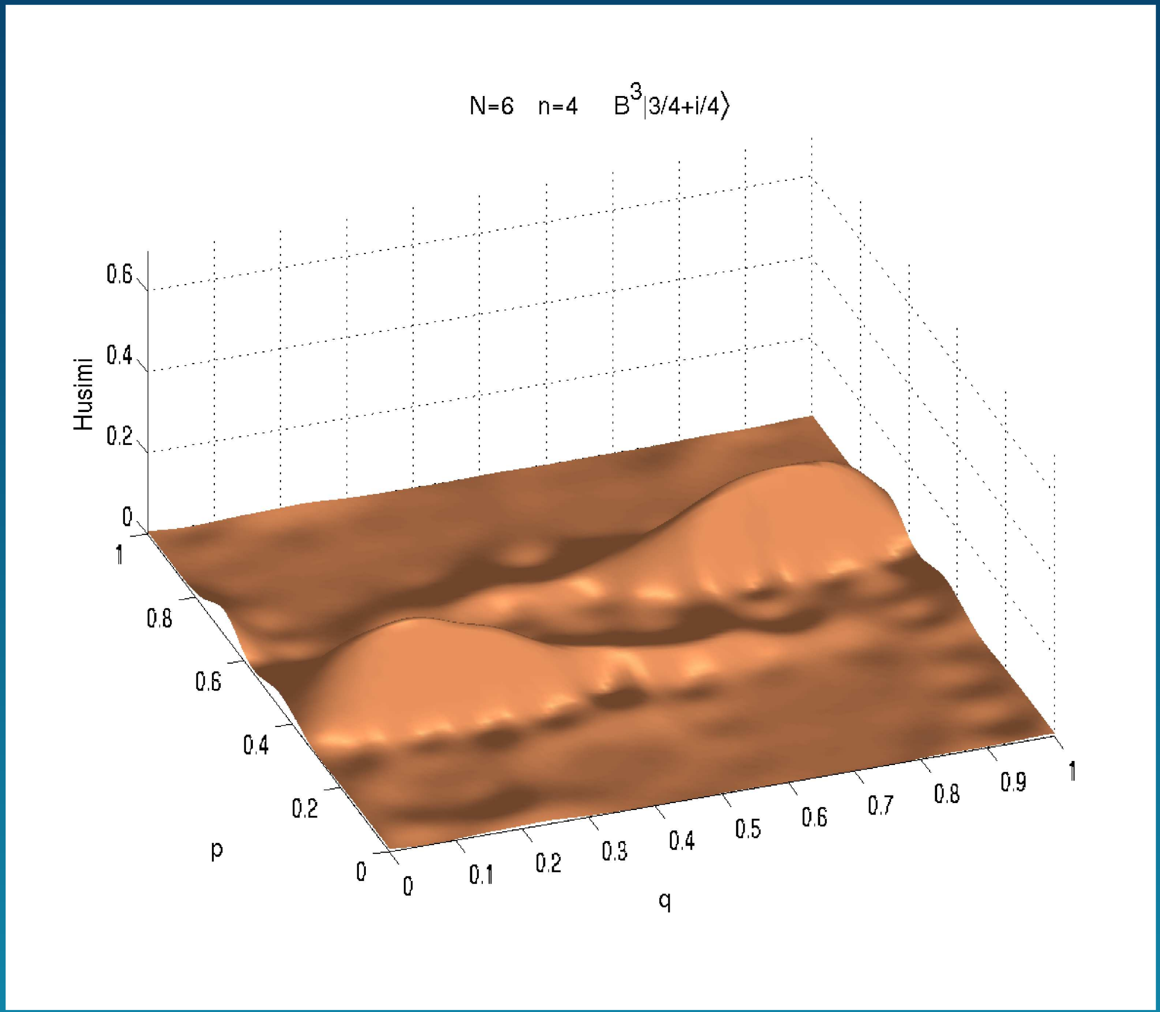

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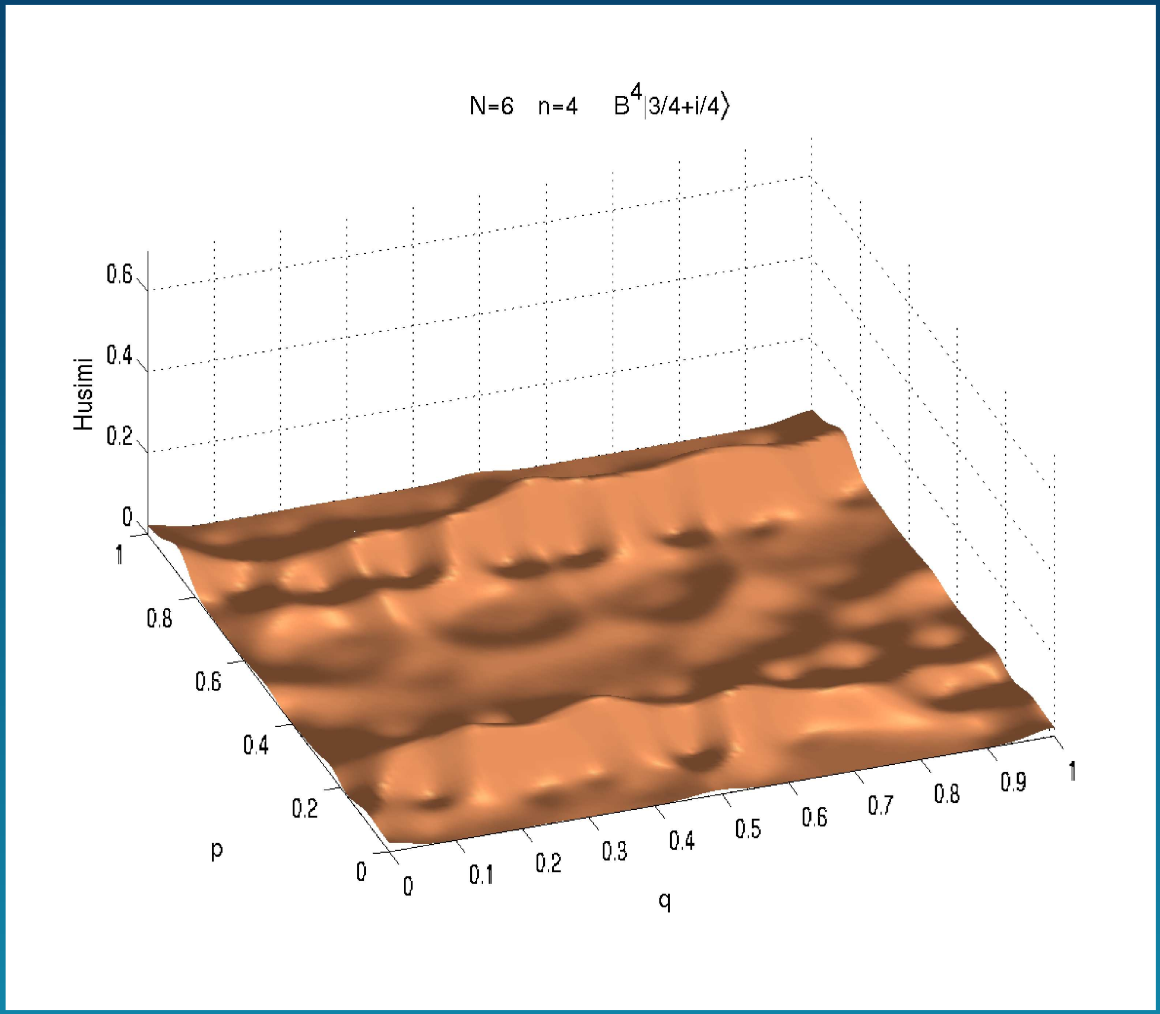

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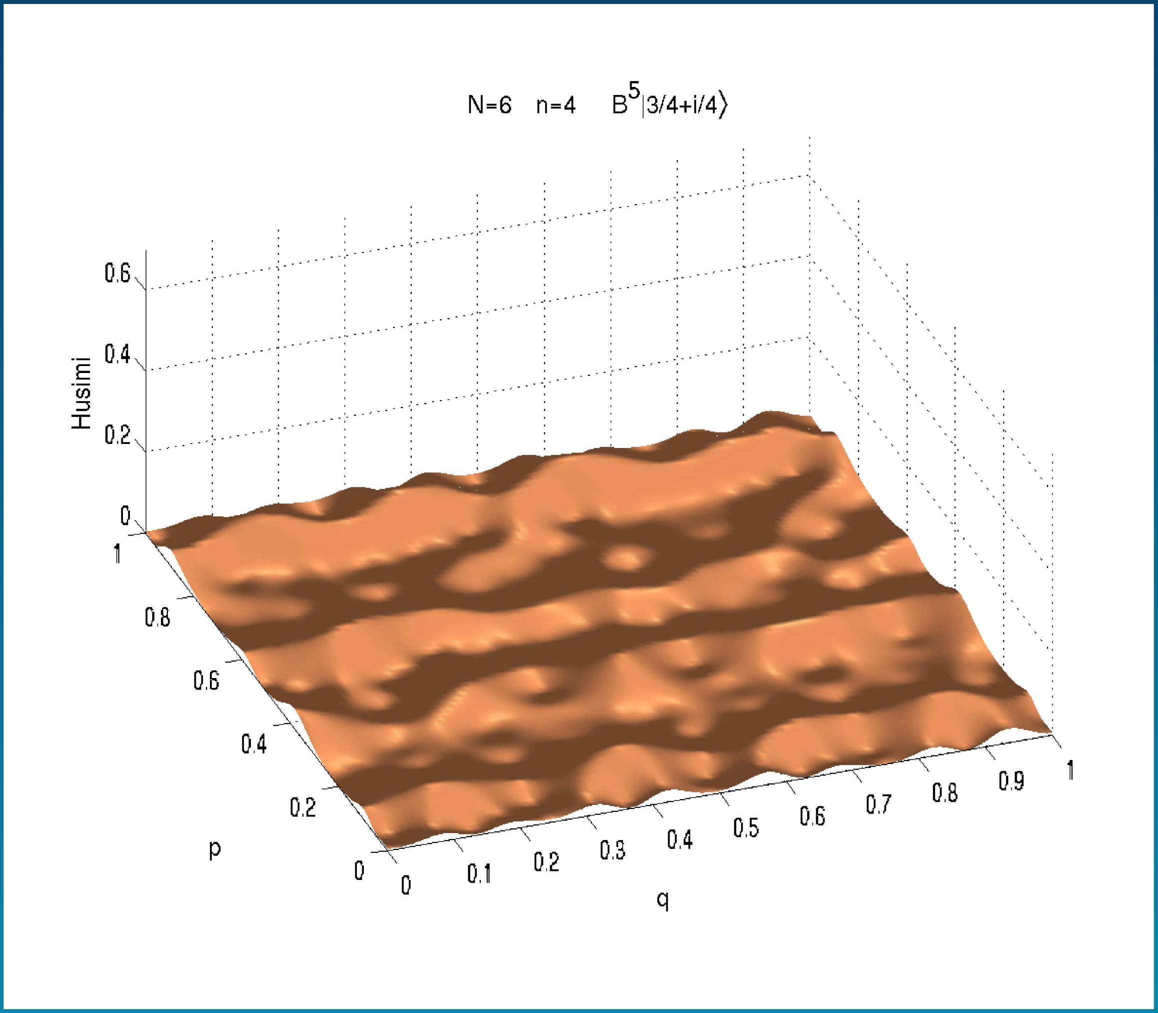

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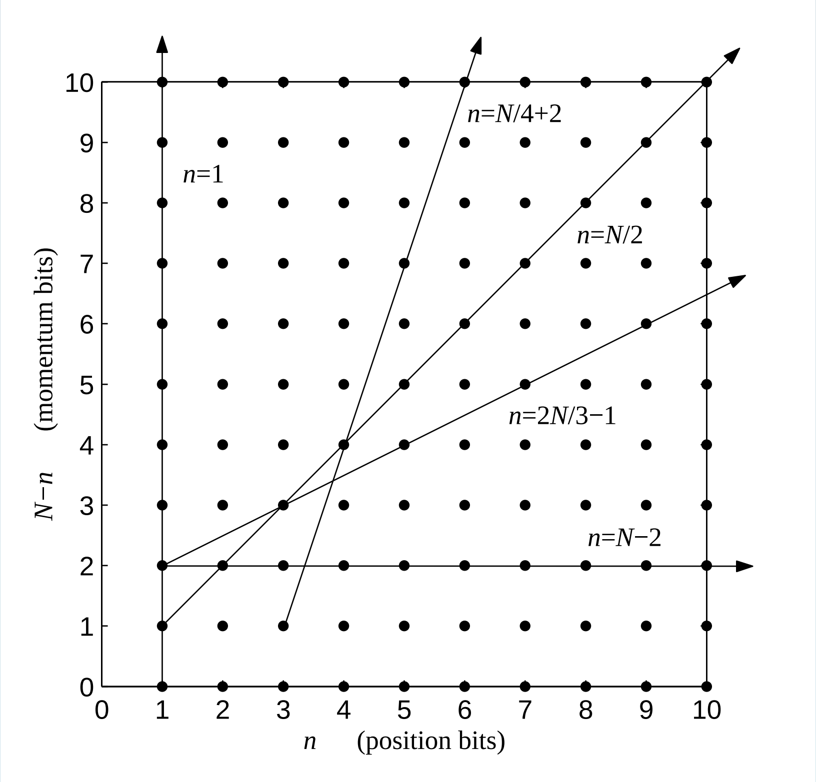

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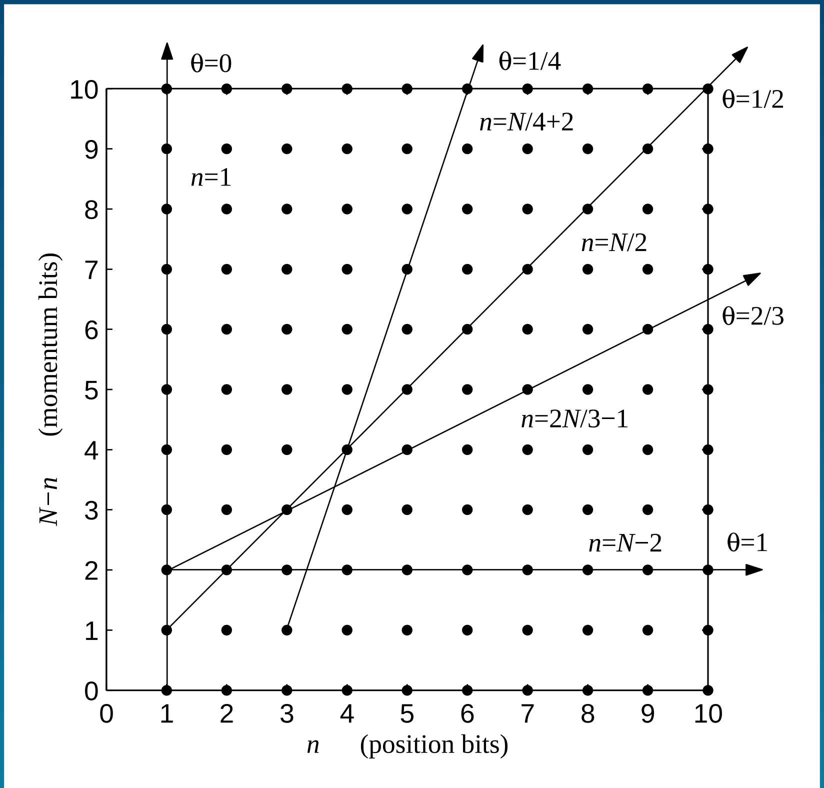

- put $n = n(N) \equiv \theta N + s$

Classical limit?

- We consider only one iteration of the map
- and assume decoherence provides classicality for long times
- We derive a semi-classical approximation for the propagator in the coherent state basis: $\langle b|\hat{B}|a\rangle \longrightarrow ?$ as $N \longrightarrow \infty$
- We need to consider the 3 cases $\theta = 0$, $0 < \theta < 1$ and $\theta = 1$ separately.

Classical limit?

- $\theta = 0$: Put $n = s \geq 1$ and define $S = 2^s$

$$
\begin{aligned}
\langle b|\hat{B}|a\rangle \quad = \quad & SD^{-3/2} \sum_{\mu,\nu=-\infty}^{\infty} \sum_{x_1=0}^{1} \sum_{j,k=0}^{D/S-1} \sum_{l=0}^{2D/S-1} \sum_{m=0}^{S/2-1} \\
& \exp\Bigg[-\frac{\pi D}{2}\Big(|a|^2 + |b|^2 - a^2 - b^{*2}\Big) + i\pi(\mu - \nu) \\
& \qquad - \pi D\Big(q_k + x_1/2 + (Dq_m - 1/2)/S - a + \mu\Big)^2 \\
& \qquad - \pi D\Big(q_l + 2(Dq_m - 1/2)/S - b^* + \nu\Big)^2 \\
& \qquad + i\pi SD\Big(q_j q_l + x_1 q_l/S - 2q_j q_k\Big)\Bigg]
\end{aligned}
$$

Classical limit?

- Now use variants of the Poisson summation formula to replace sums with integrals

$$
\begin{aligned}
\langle b|\hat{B}|a\rangle \quad = \quad & SD^{3/2} \sum_{\substack{\mu,\nu,\alpha \\ \beta,\gamma=-\infty}}^{\infty} \sum_{x_1=0}^{1} \sum_{m=0}^{S/2-1} \int_0^{1/S} dx \int_0^{1/S} dy \int_0^{2/S} dz \\
& \exp\Bigg[-\frac{\pi D}{2}\left(|a|^2+|b|^2-a^2-b^{*2}\right)+i\pi(\mu-\nu-\alpha-\beta-\gamma) \\
& -\pi D\Big(y+x_1/2+m/S-a+\mu\Big)^2-\pi D\Big(z+2m/S-b^*+\nu\Big)^2 \\
& +i\pi SD\Big(xz+x_1 z/S-2xy\Big)+2i\pi D\left(x\alpha+y\beta+z\gamma\right)\Bigg]
\end{aligned}
$$

Classical limit?

- Now use the method of steepest descents to make a saddle point approximation on the triple integral. And with a little algebra

$$
\begin{aligned}
\langle b|\hat{B}|a\rangle \quad = \quad & \sqrt{\frac{4}{5}}\exp\Big[-\frac{\pi D}{5}\Big\{\big(2a_1-b_1-\lfloor 2a_1\rfloor\big)^2+\big(a_2-2b_2+\lfloor 2a_1\rfloor\big)^2 \\
& +i\big(3a_1a_2+3b_1b_2+4a_1b_2-4a_2b_1\big) \\
& -2i\lfloor 2a_1\rfloor\big(a_1+2b_1+2a_2+b_2-\lfloor 2a_1\rfloor/2\big)\Big\}\Big]+o(1)
\end{aligned}
$$

- where $a = a_1 + ia_2$ and $b = b_1 + ib_2$

- The semi-classical propagator is $O(1)$ only when (b_1, b_2) is the iterate of (a_1, a_2) under the action of the classical baker's map

Classical limit?

- The semi-classical propagator can be rewritten in the Van Vleck form

$$\langle b|\hat{B}|a\rangle \quad = \quad \sqrt{\frac{\partial^2 W}{\partial a \partial b^*}} \exp\Big[\pi D W(b^*, a)\Big] \exp\Big[- \pi D \left(|a|^2 + |b|^2\right)/2\Big] + o(1)$$

- where $W(b^*, a)$ is a generating function for the classical baker's map rewritten in terms of the complex variables a and a^*

- $0 < \theta < 1?$ More complicated derivation, but same end result.

- $\theta = 1?$ Something different occurs.

Classical limit?

- 2 different classical limits:

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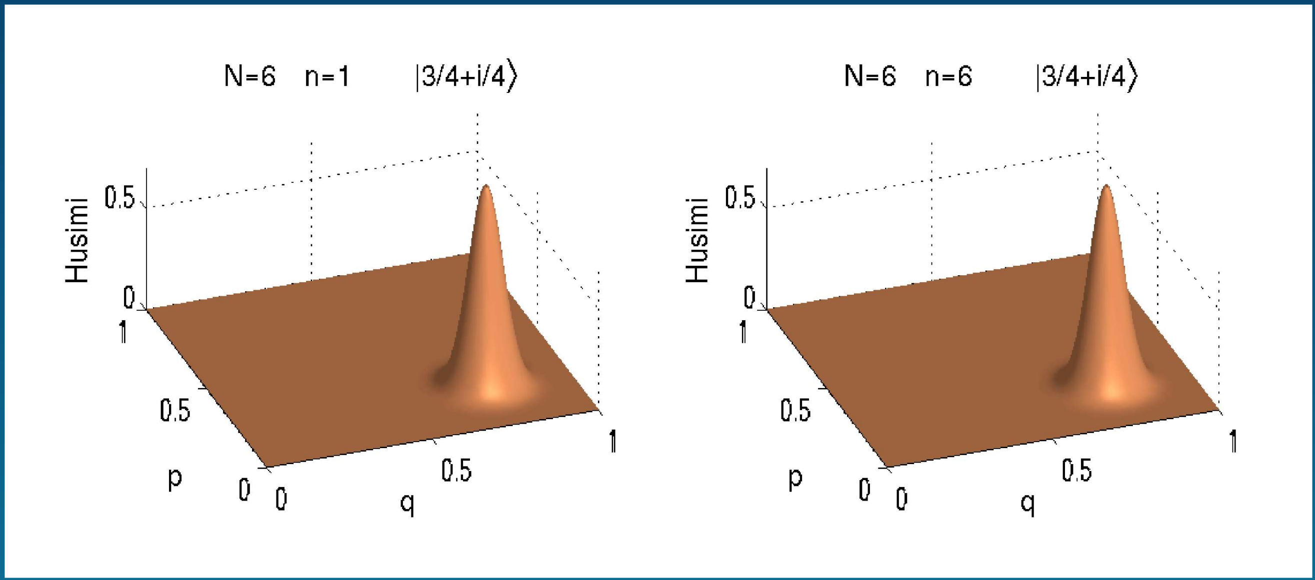

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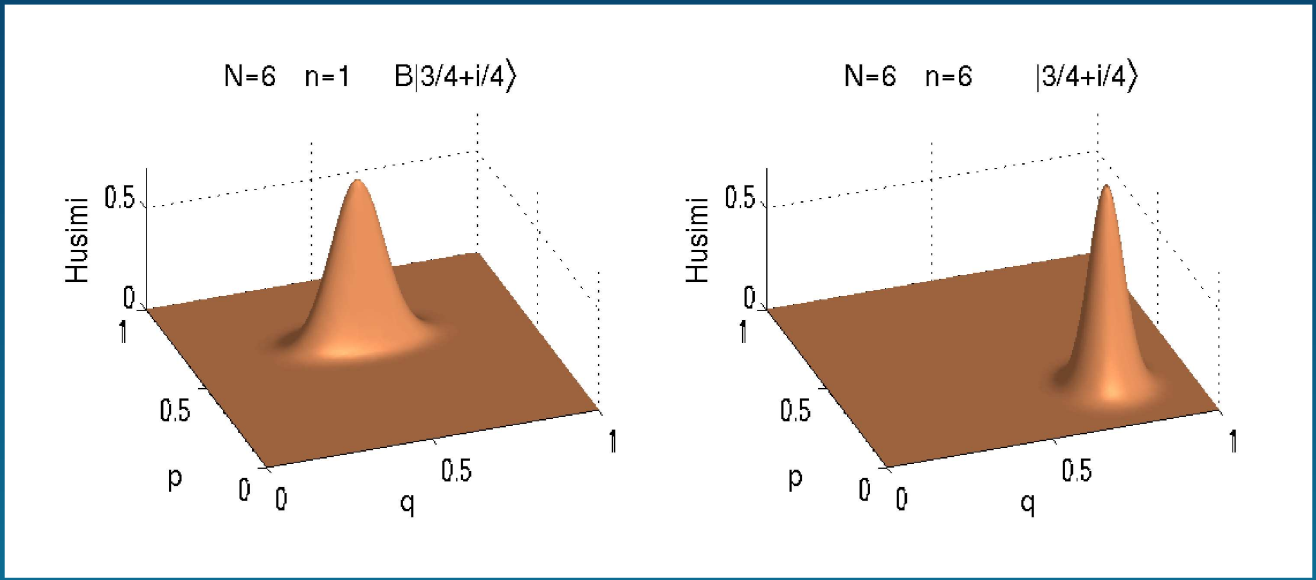

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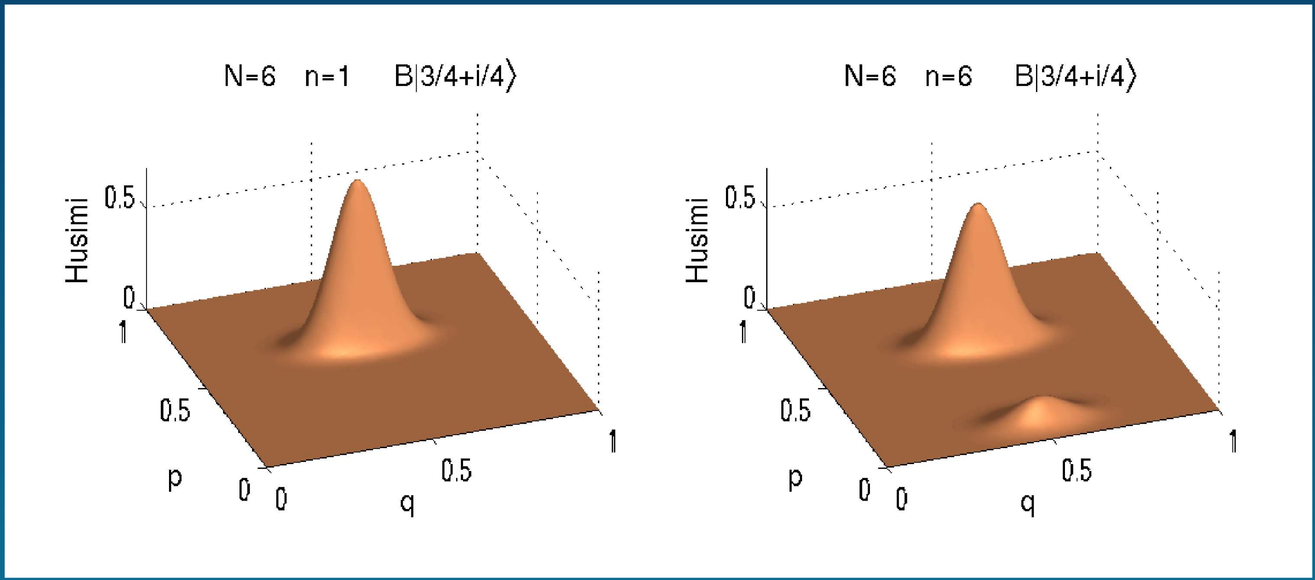

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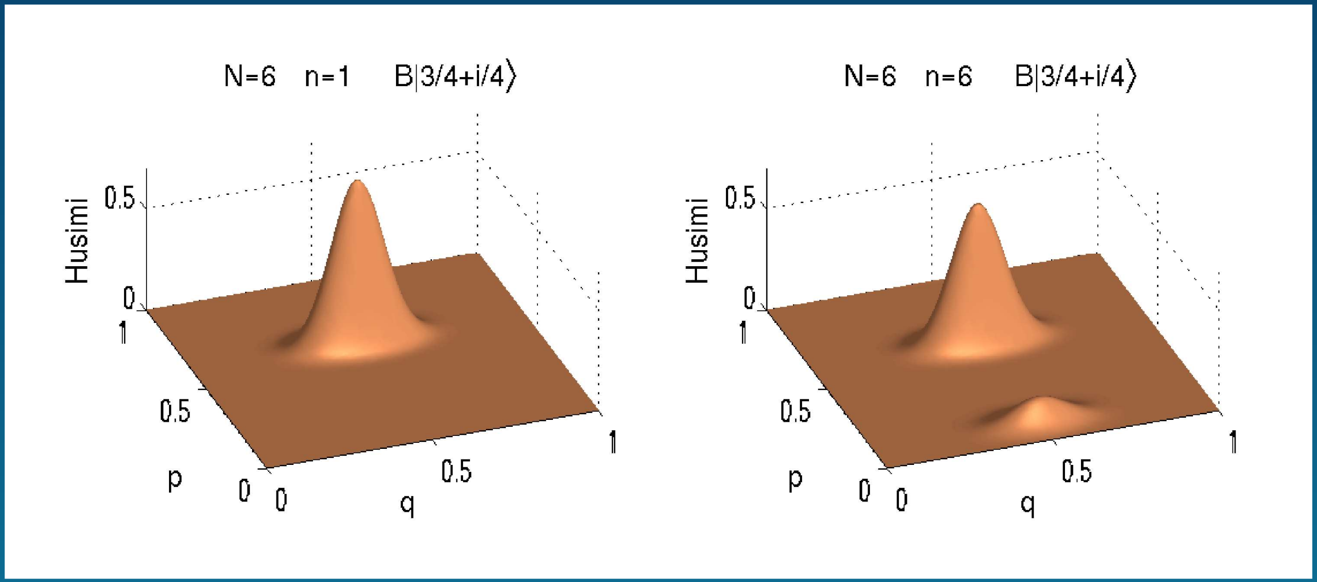
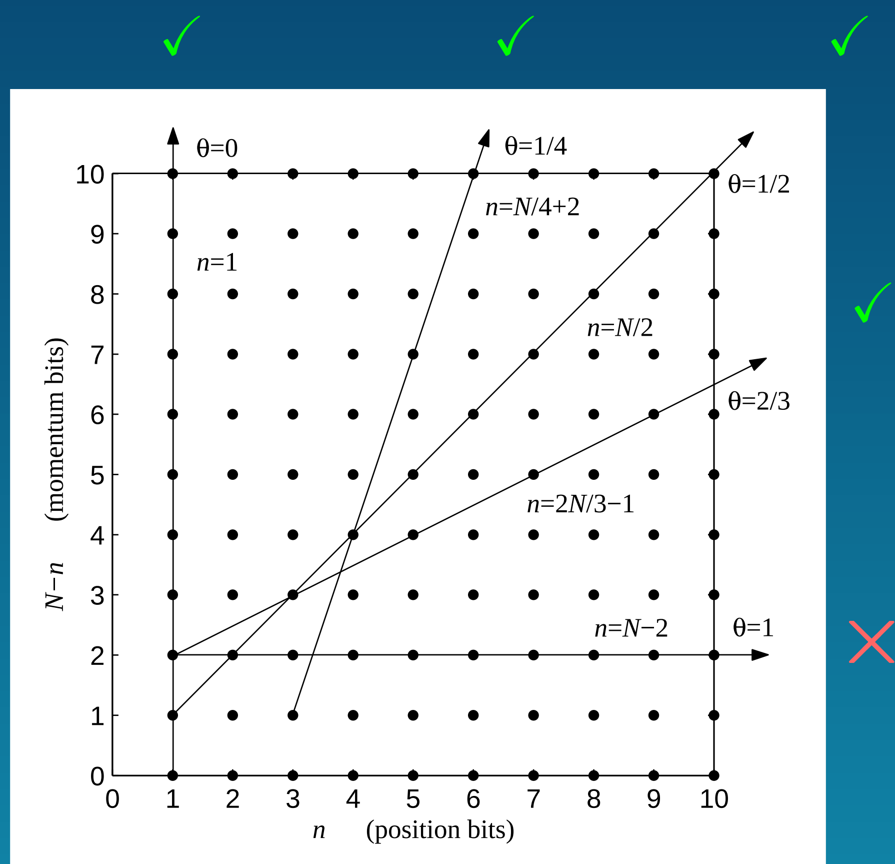


- Increasing the number of position bits while keeping constant the number of momentum bits creates additional ‘humps’ in the wrong places.
- A stochastic mapping is implied in the classical limit ?

Classical limit?

Conclusion

The Schack-Caves quantum baker's maps have the proper classical limit, provided the number of momentum bits is allowed to approach infinity in this limit.



Entangling power

- The quantum baker's map with $n = N$, where the number of momentum bits is fixed at 0, is nonentangling, taking product states to product states.
- What is the entangling power of the remaining quantum baker's maps?
- We might expect such maps to be good at creating random states in Hilbert space.
- To calibrate our investigation, we first need to calculate the expected value of entanglement for random pure states.

Entangling power

- When a bipartite quantum system is in an overall pure state the unique measure of entanglement is given by the subsystem entropy.

- If we sample random pure states in $\mathcal{H}_D$ according to the unitarily invariant Haar measure then the mean linear entropy of a subsystem with dimension $\mu < D$ is

$$\langle S_L \rangle = \beta \frac{(\mu - 1)(\nu - 1)}{\mu\nu + 1} \qquad \text{(Lubkin, 1978)}$$

- the variance is

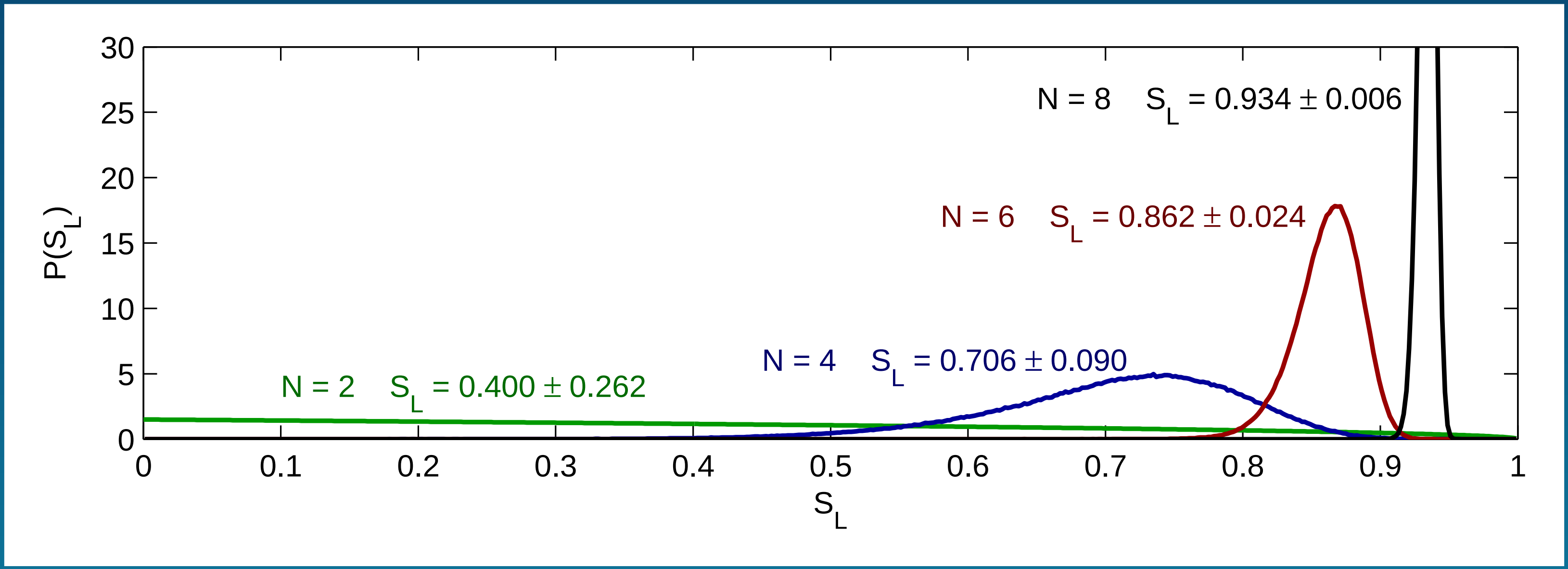
$$\langle {S_L}^2 \rangle - \langle S_L \rangle^2 = \beta^2 \frac{2(\mu^2 - 1)(\nu^2 - 1)}{(\mu\nu + 3)(\mu\nu + 2)(\mu\nu + 1)^2}$$

where

- $S_L \equiv \beta(1 - \operatorname{tr} \rho^2)$
- $0 \leq S_L \leq 1$
- $\beta \equiv \mu/(\mu - 1) \qquad (\mu \leq \nu)$
- $\mu\nu = D$

Entangling power

- For N-qubit random states, $D = 2^N$, and a partition dividing the subsystems equally $\mu = \nu = \sqrt{D}$



- Typical multi-qubit pure states have HIGH levels of bipartite entanglement.
- The entanglement distributions are highly localized about the mean.

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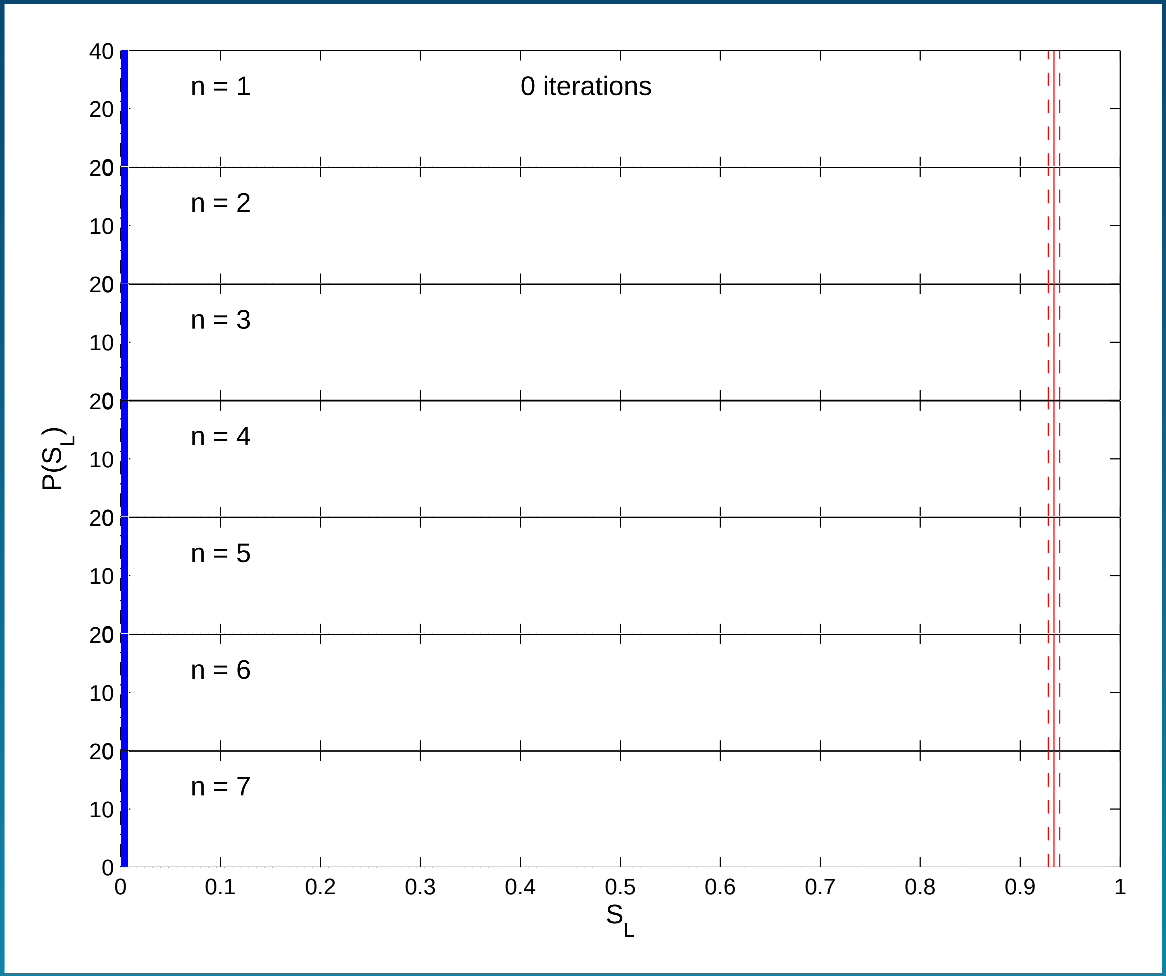

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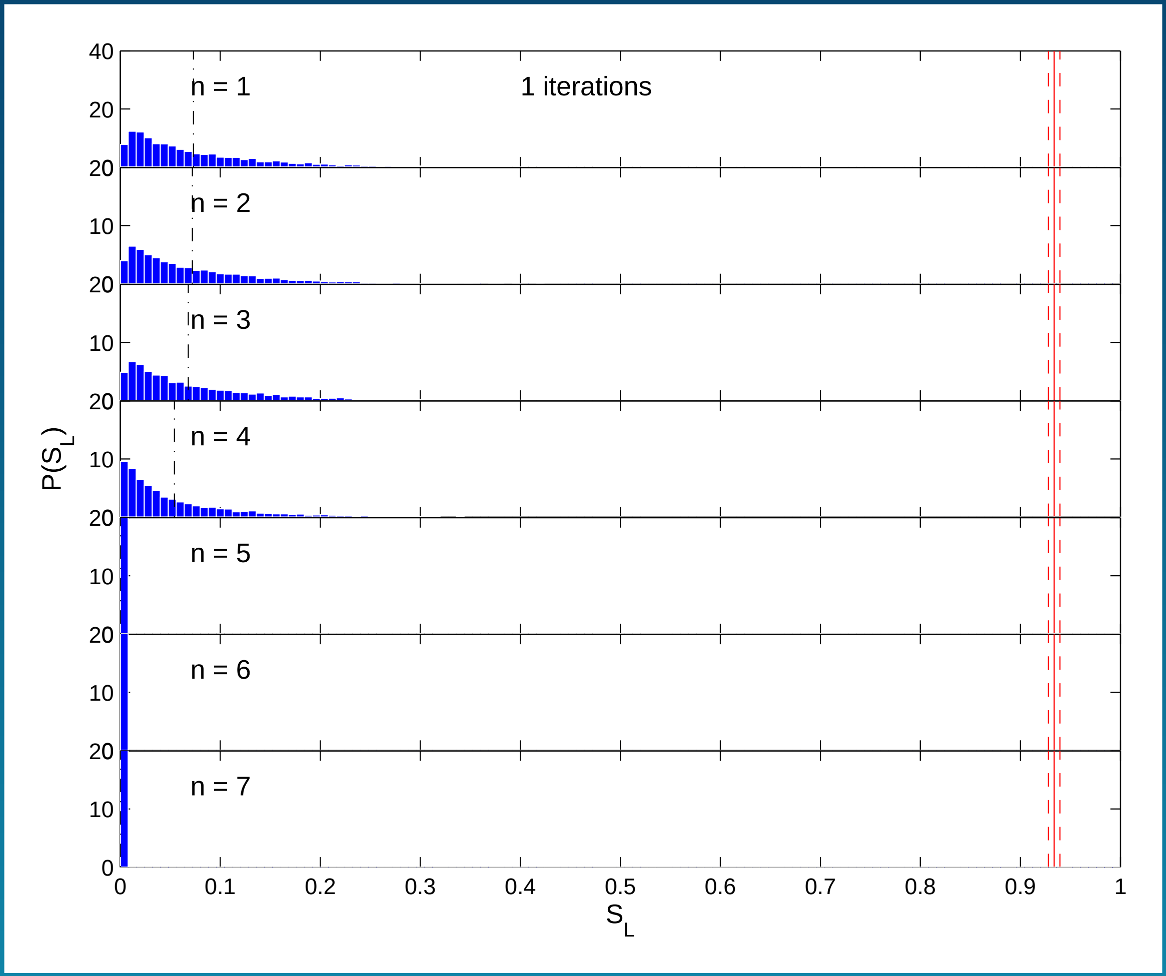

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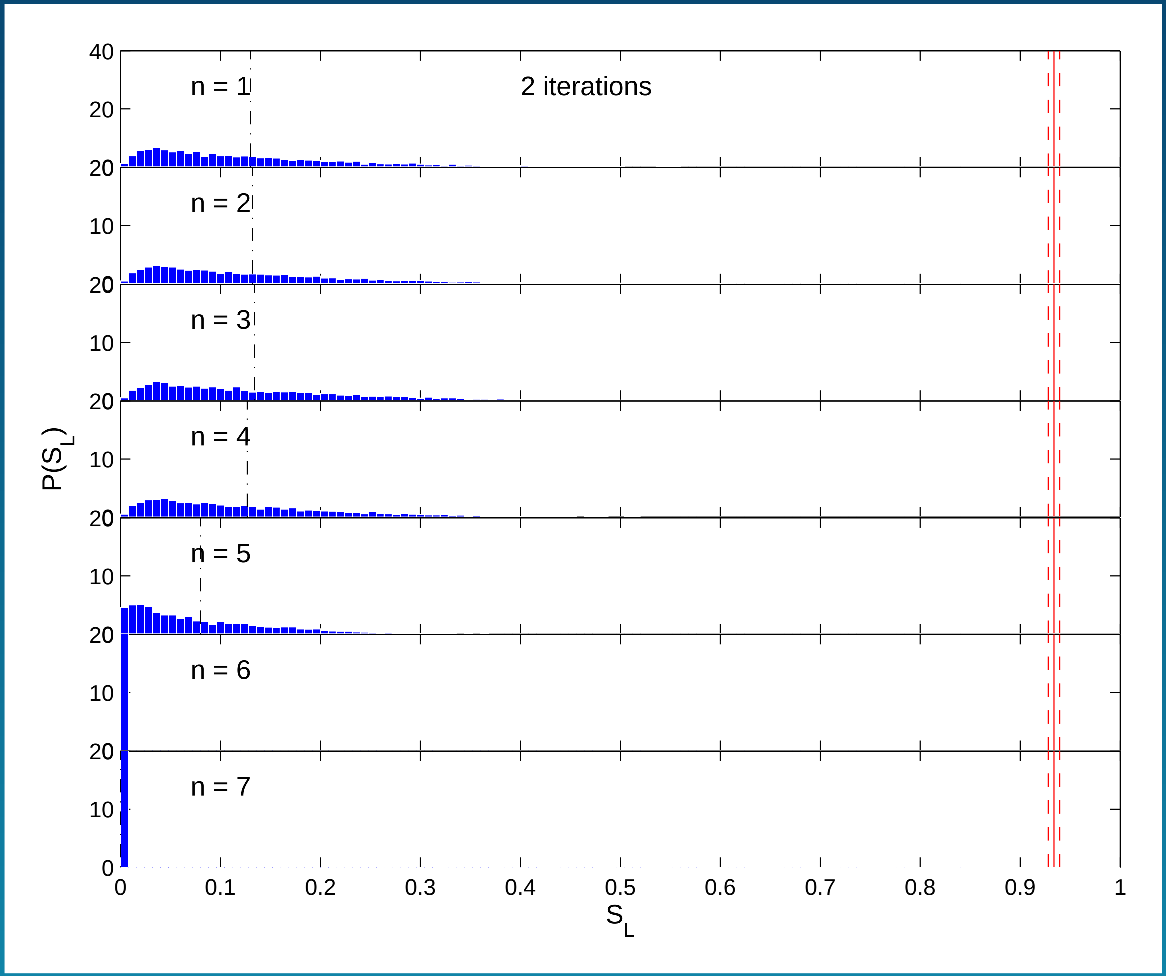

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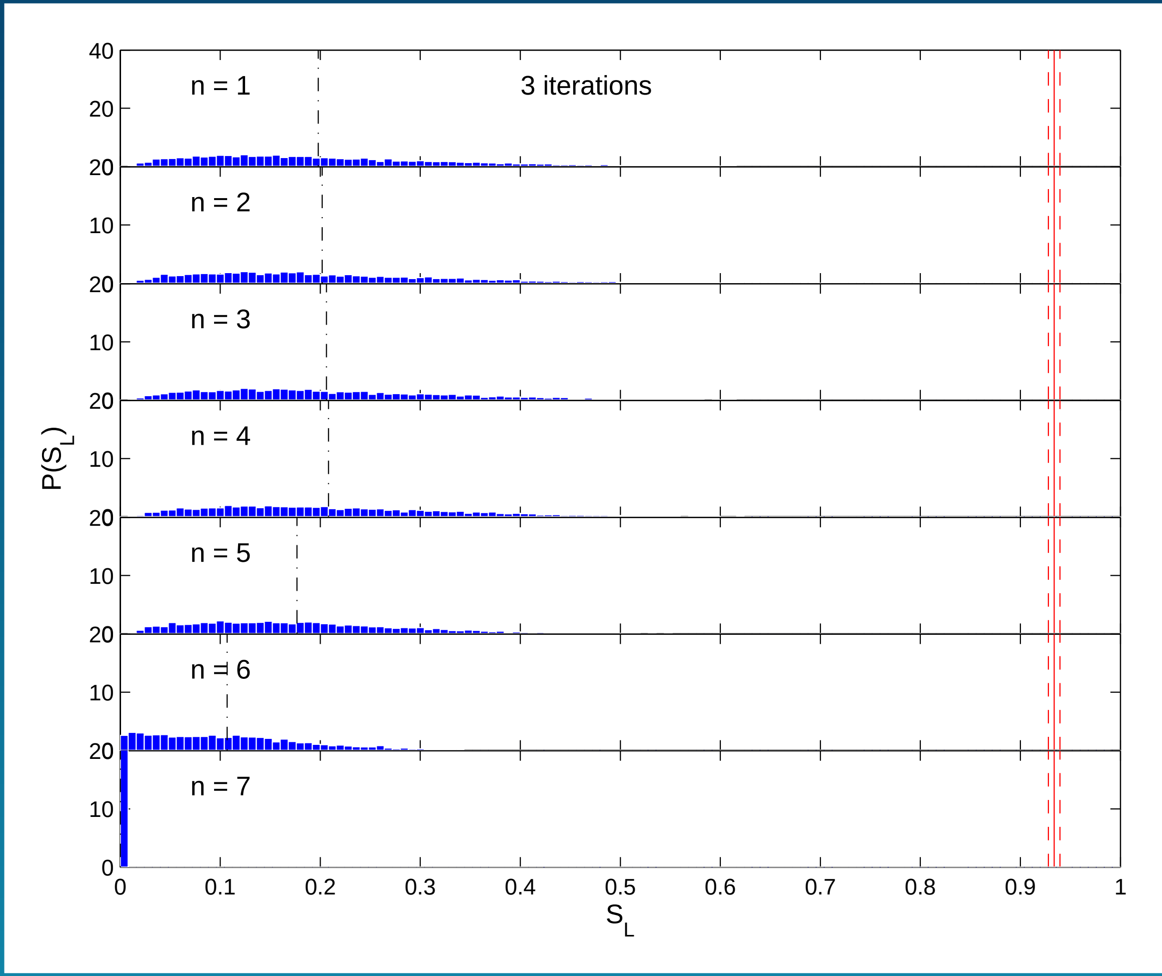

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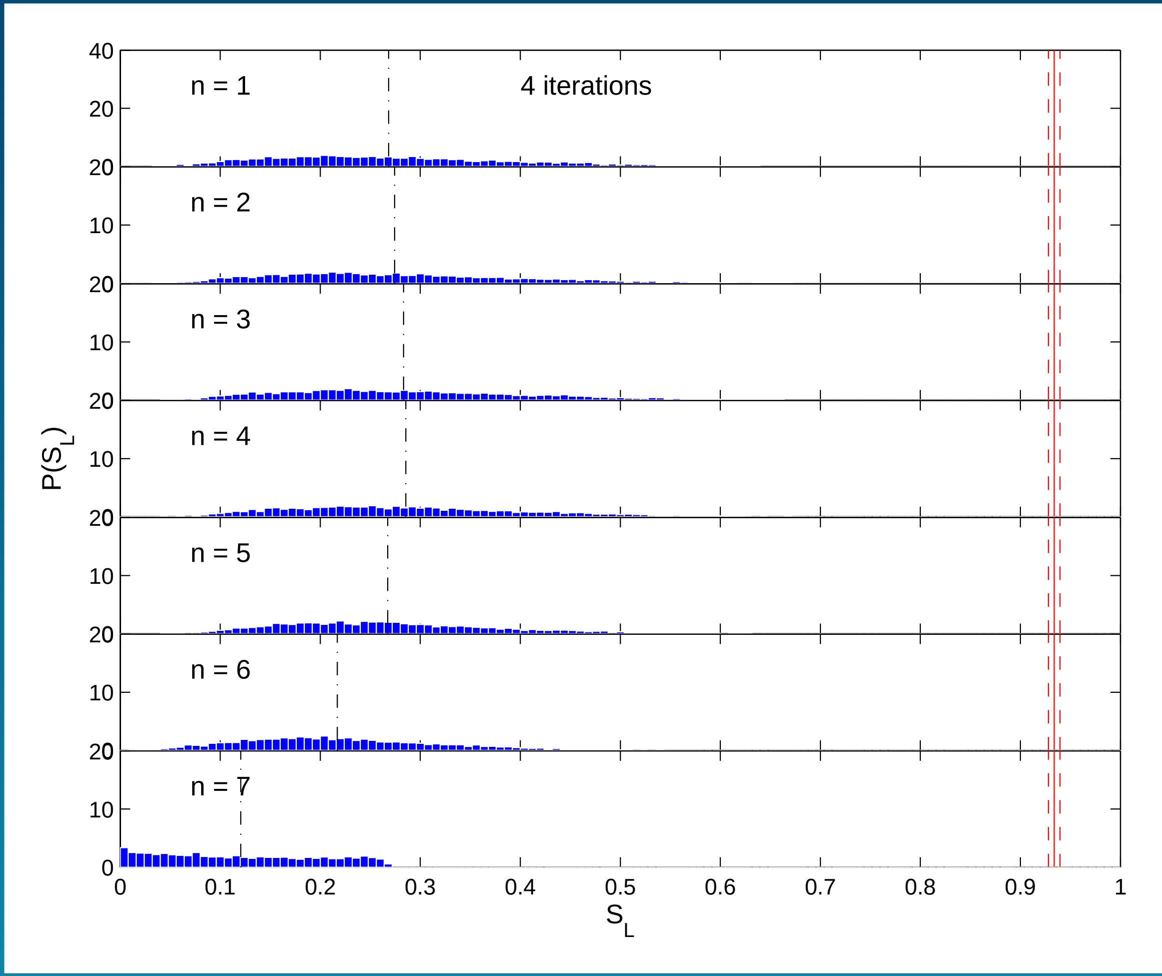

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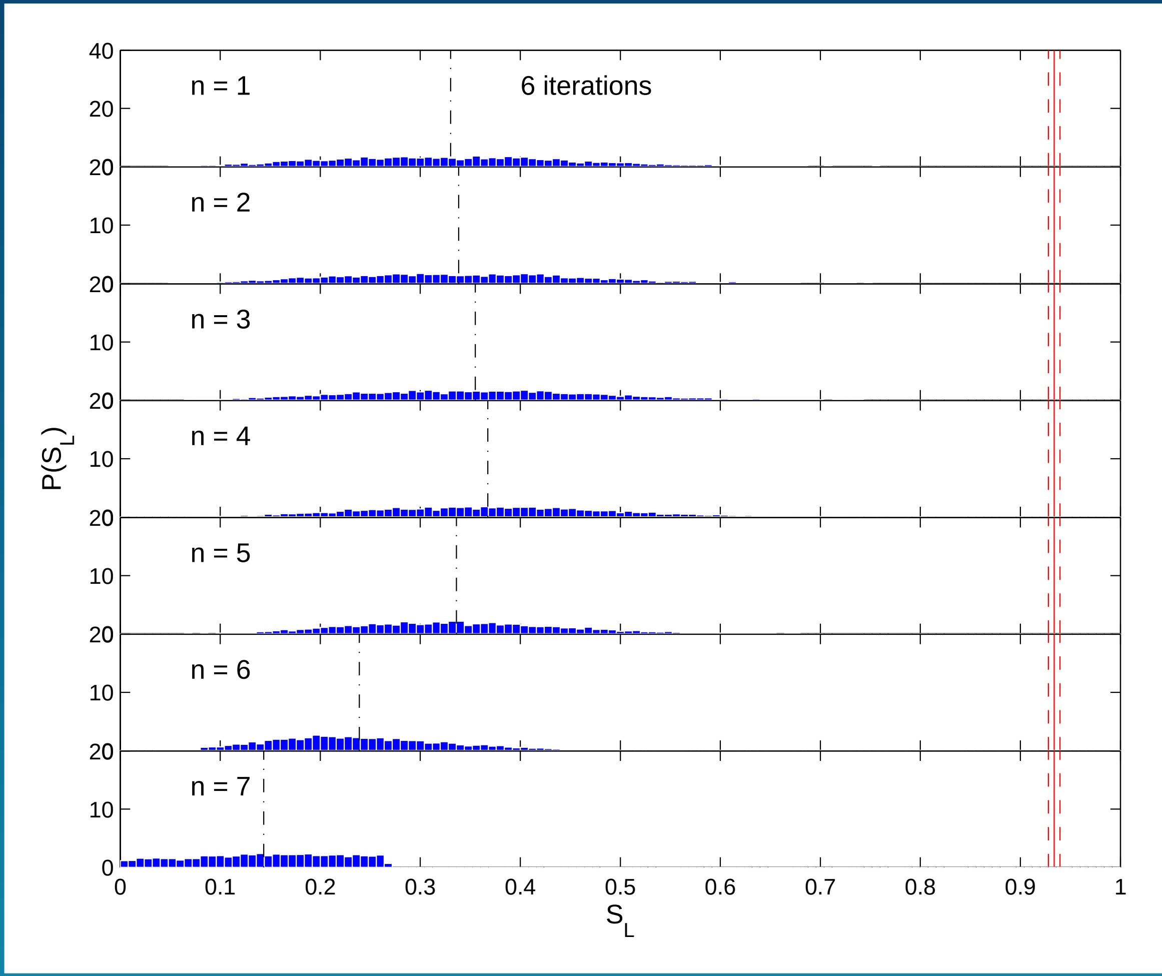

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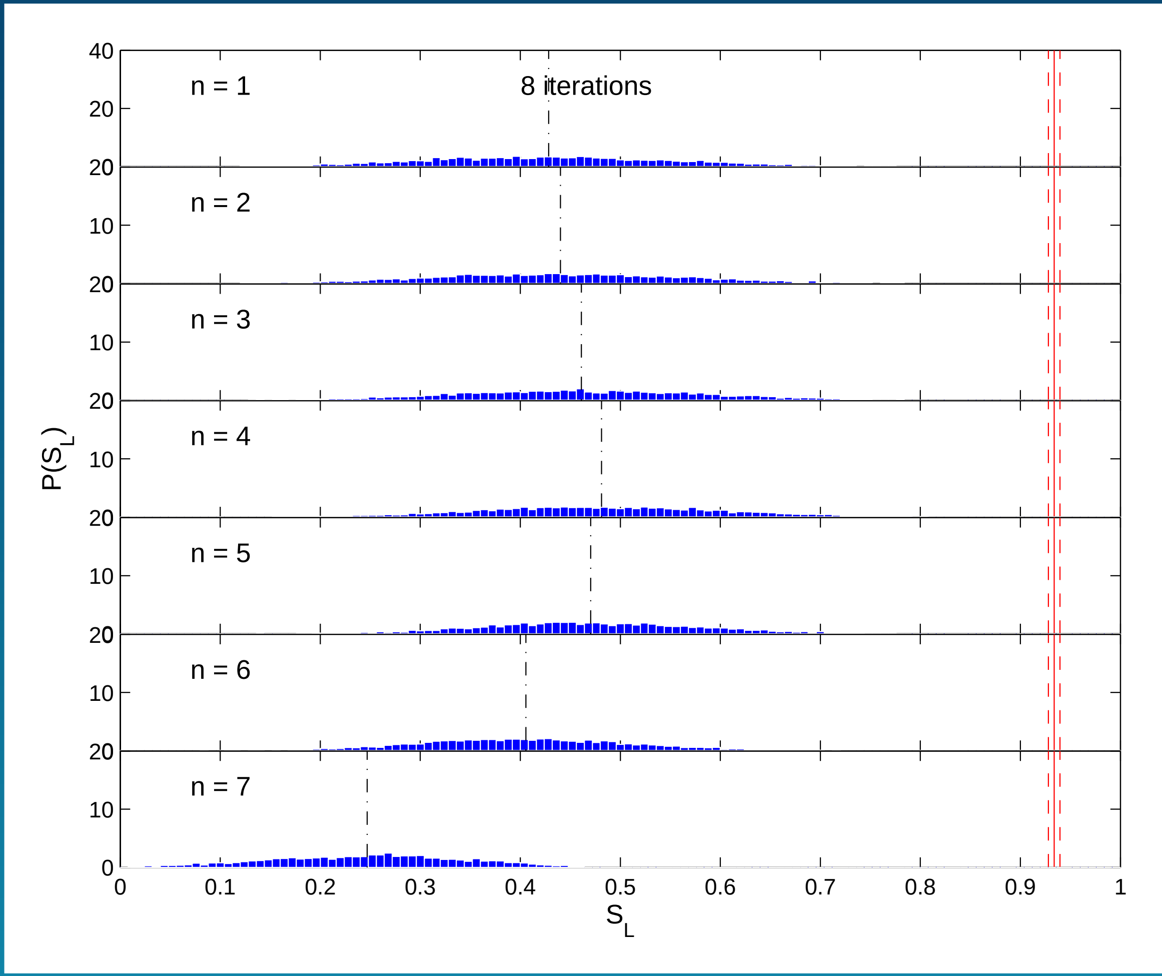

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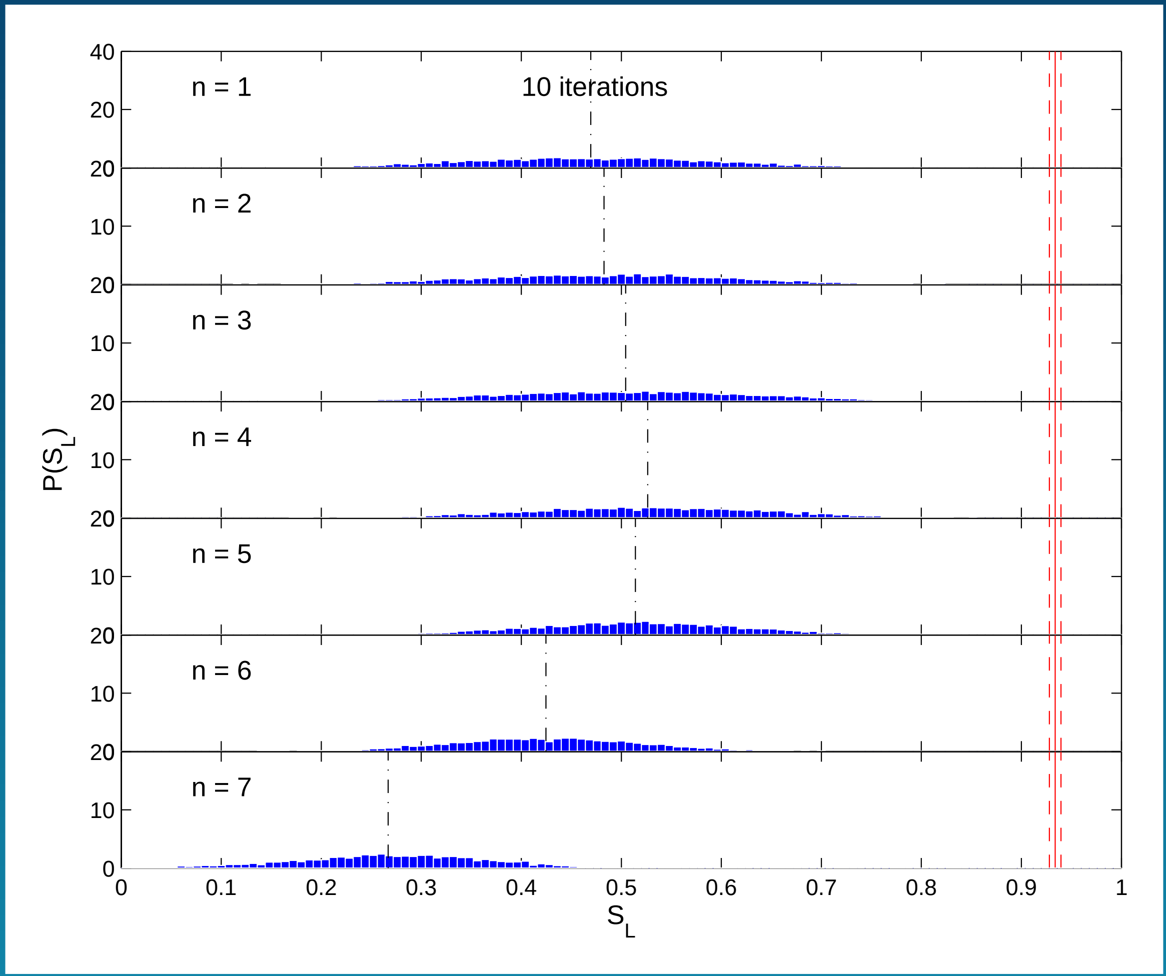

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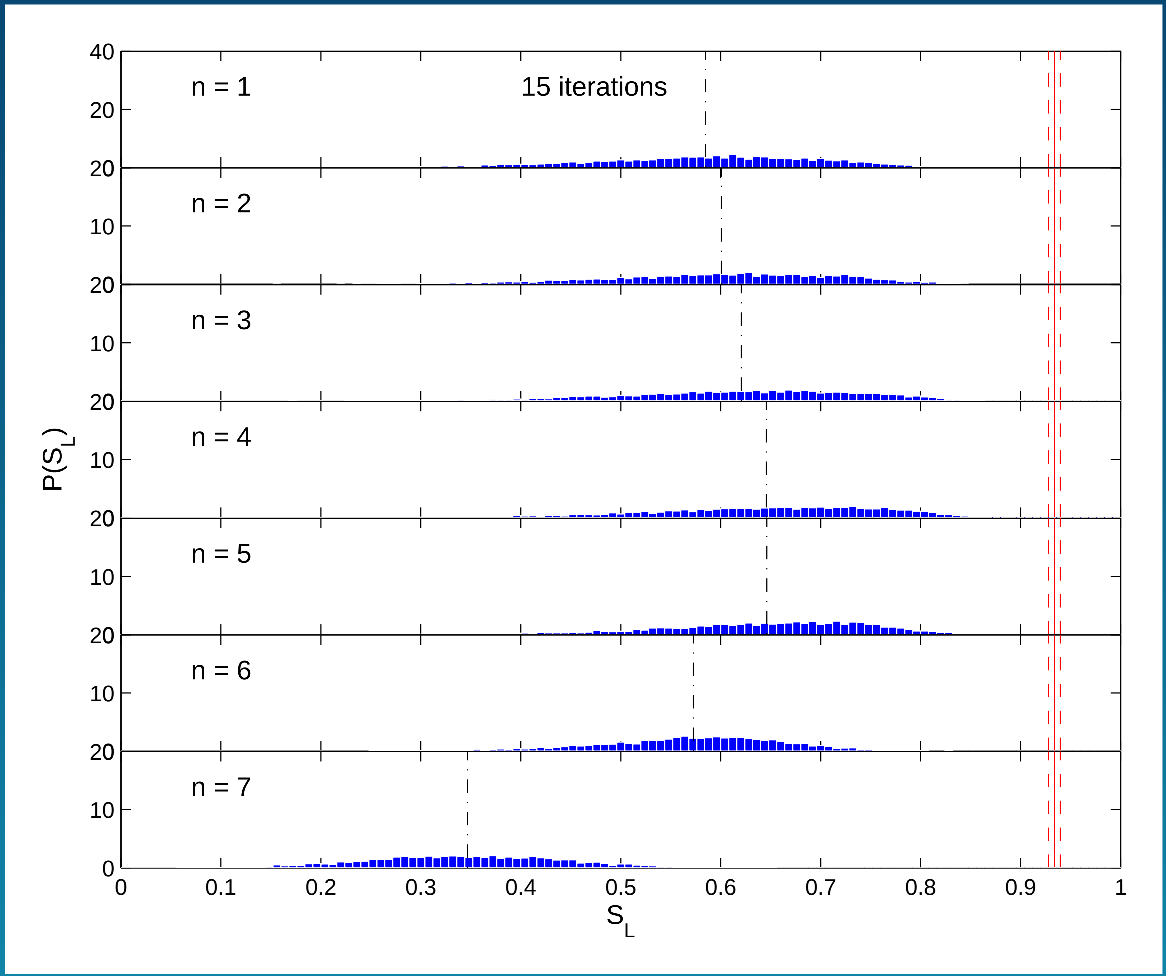

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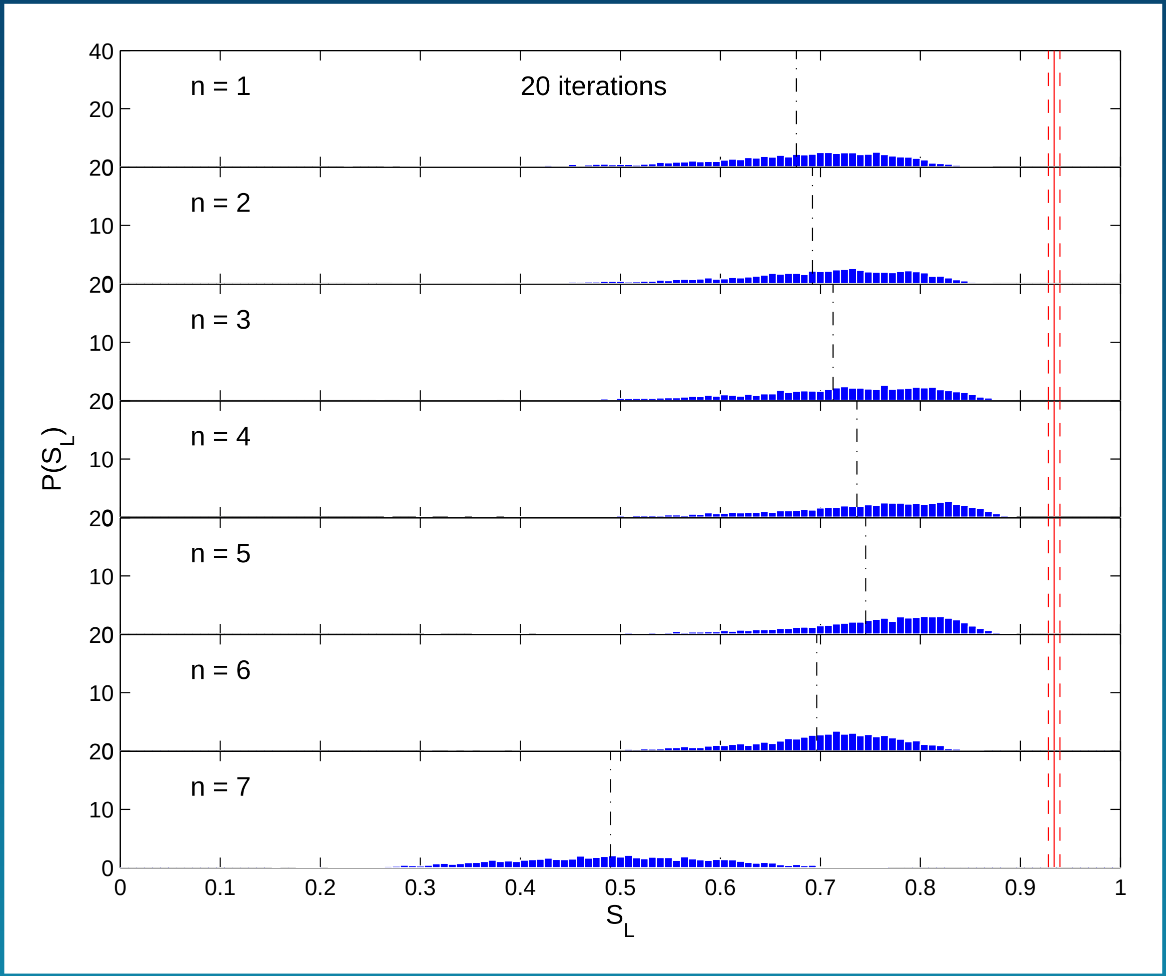

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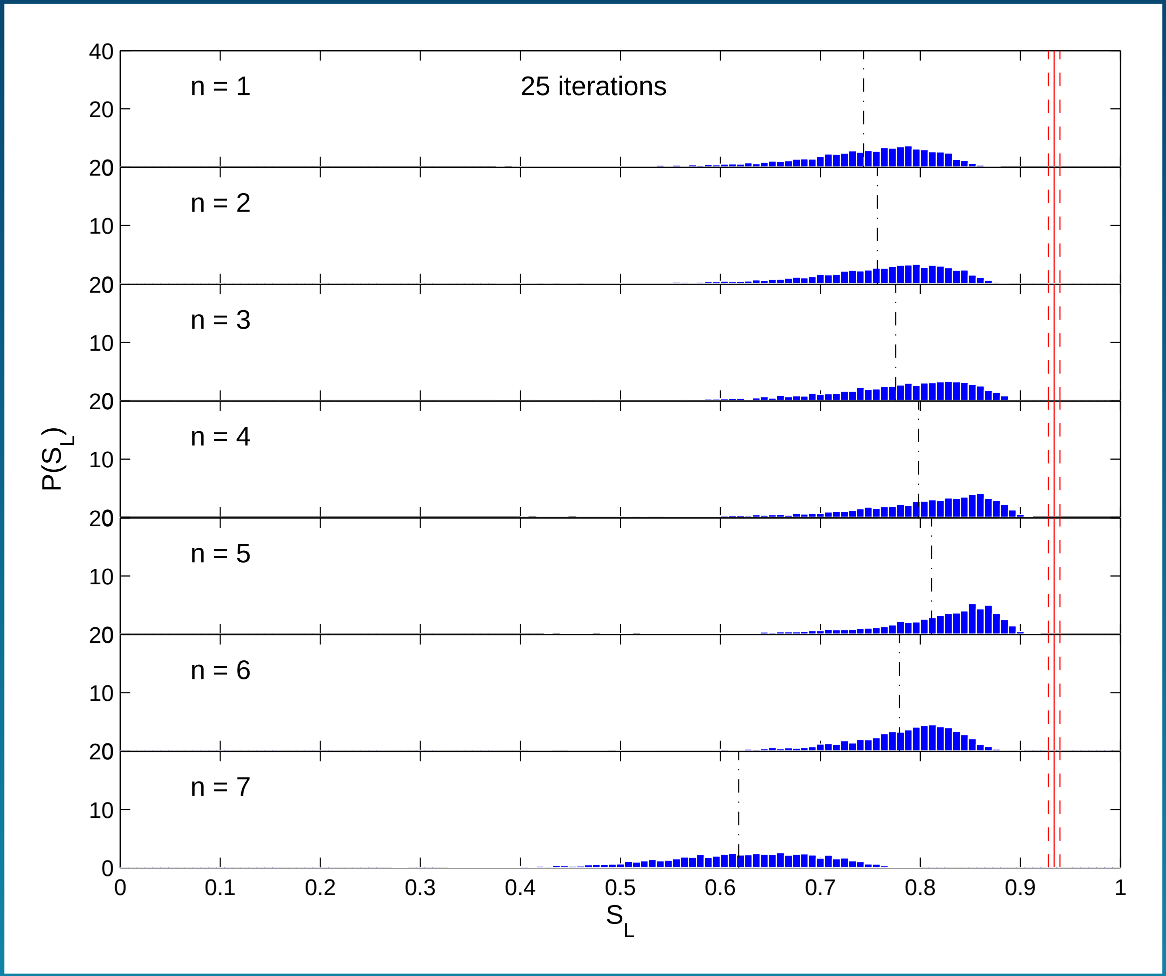

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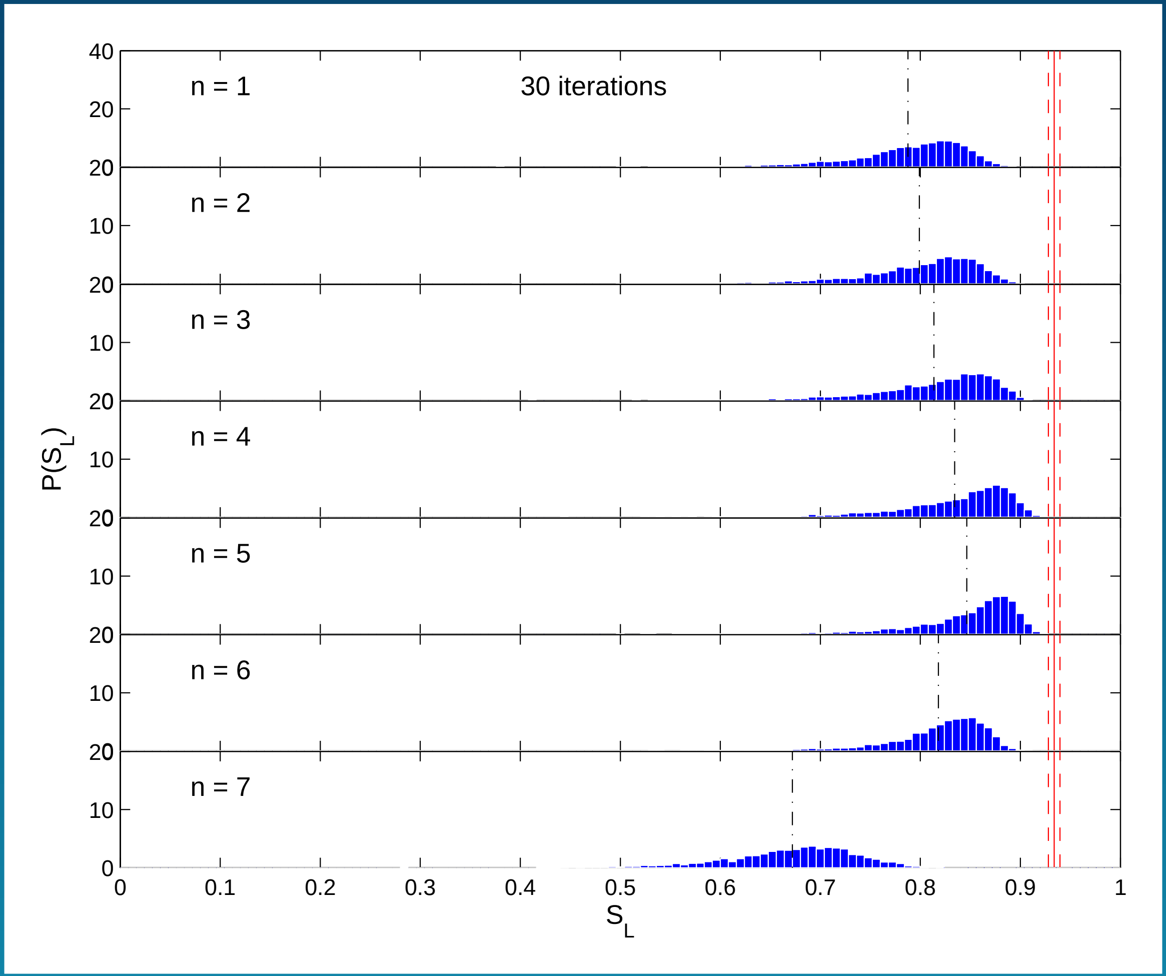

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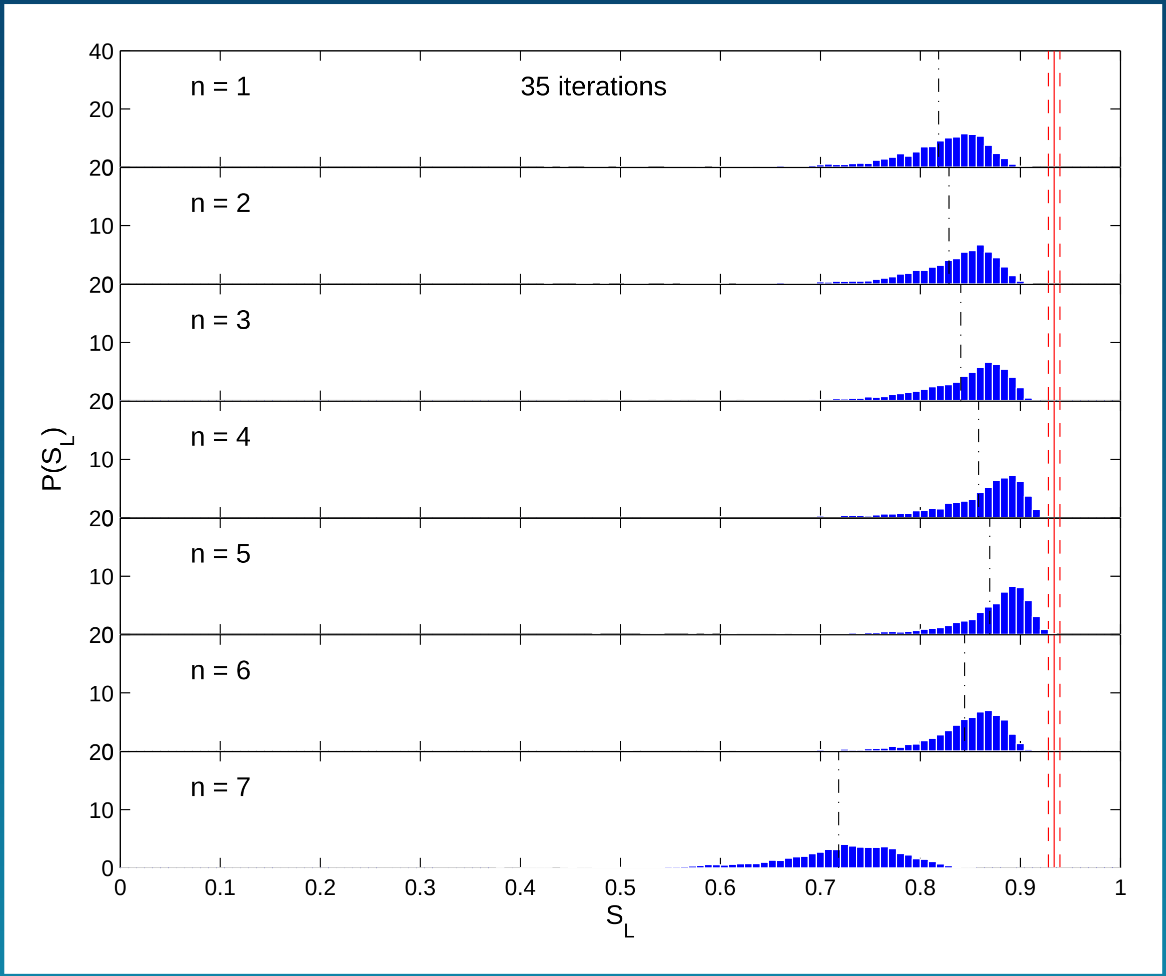

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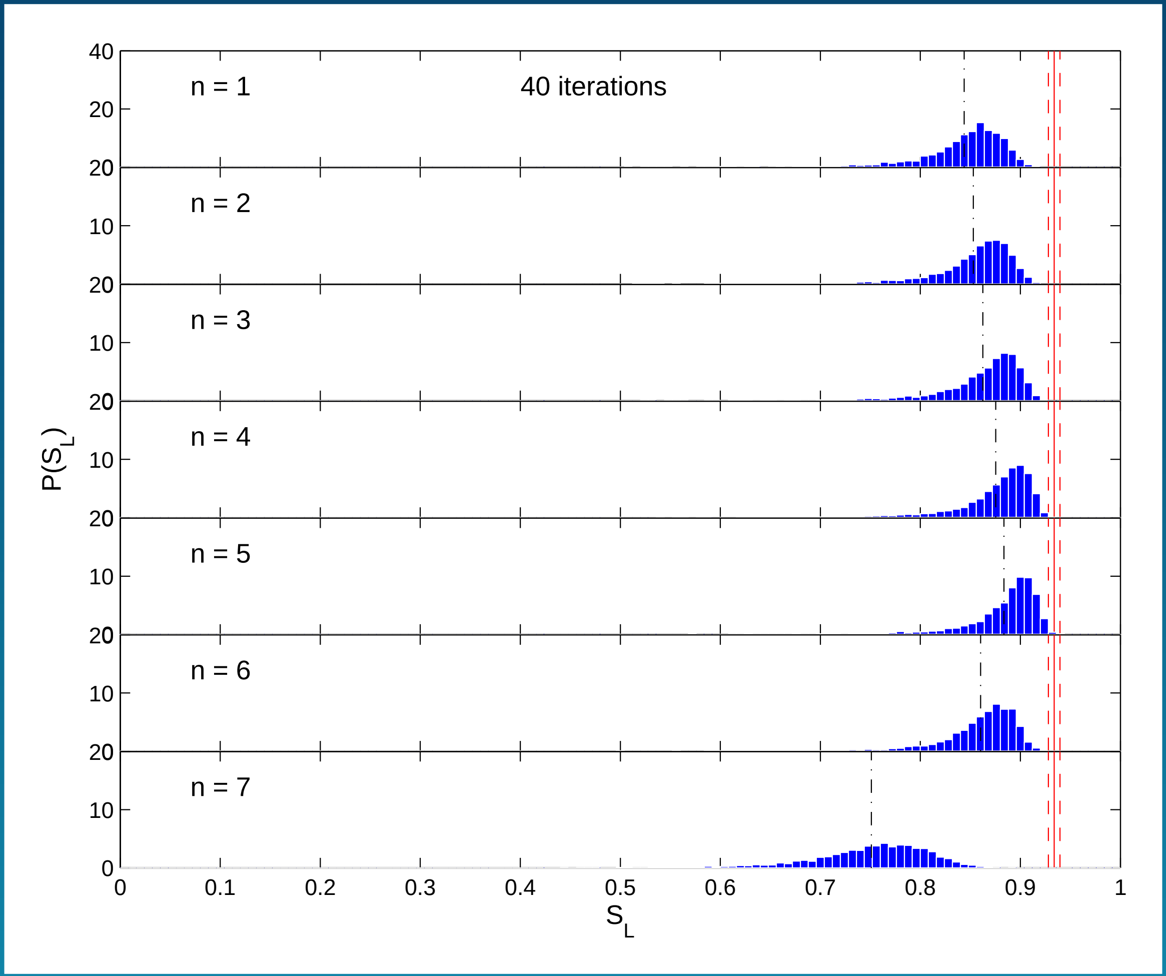

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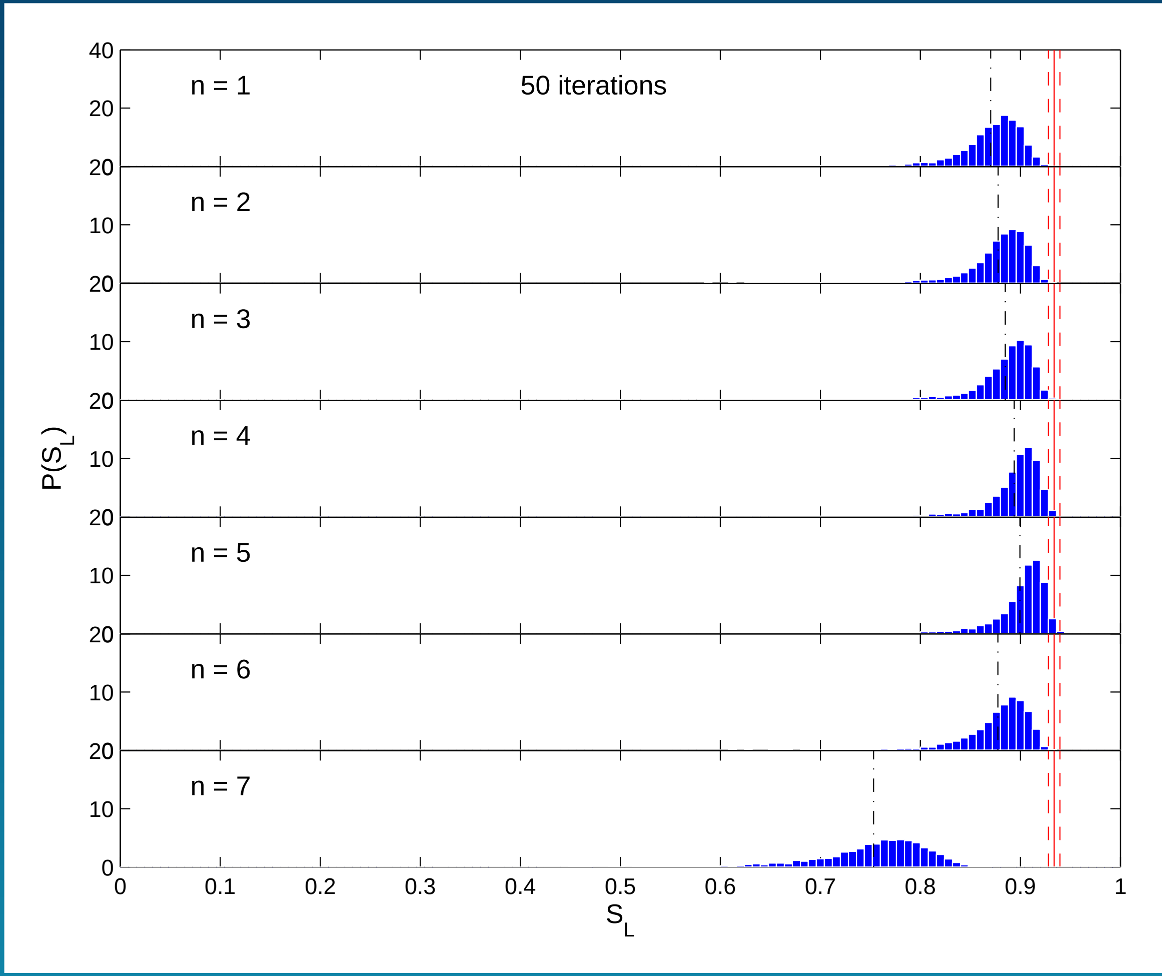

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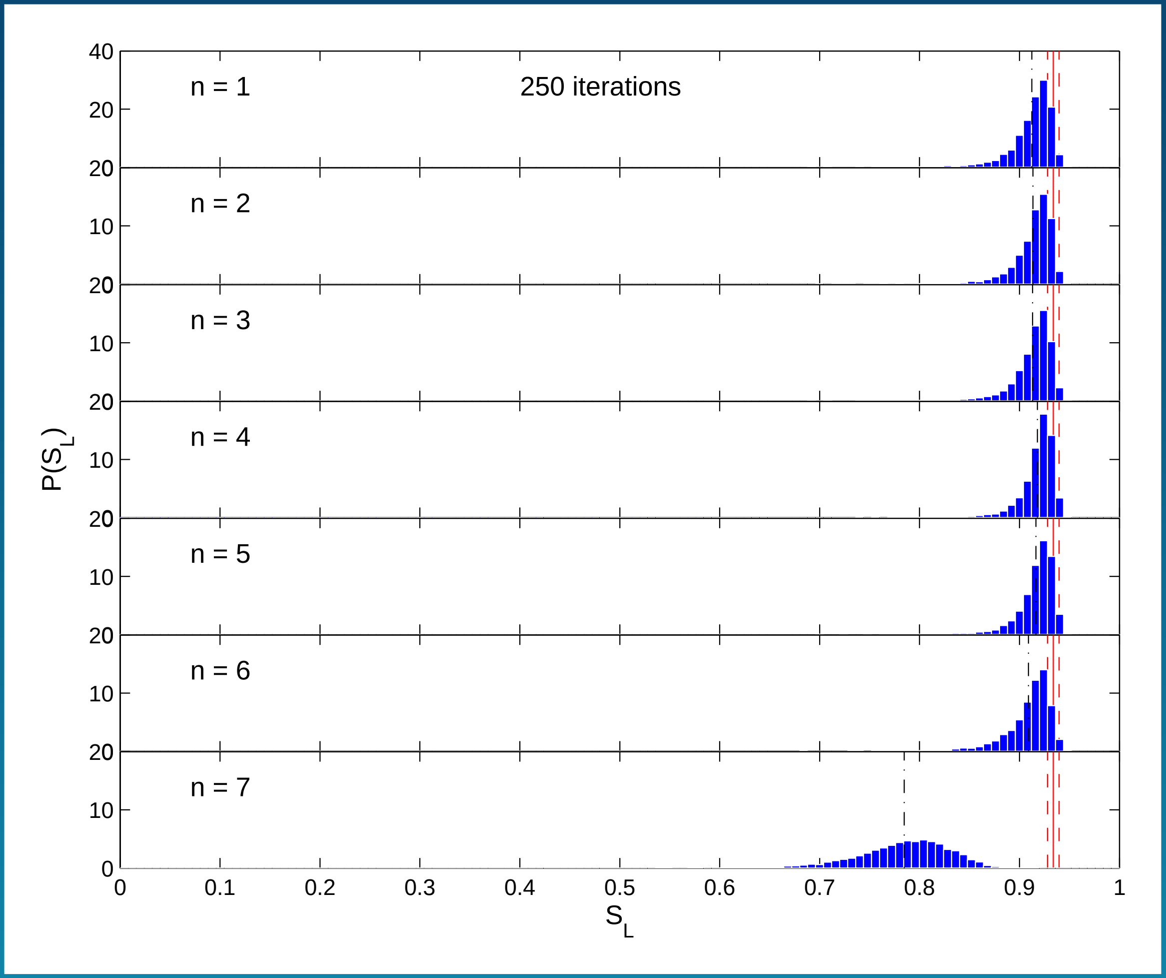

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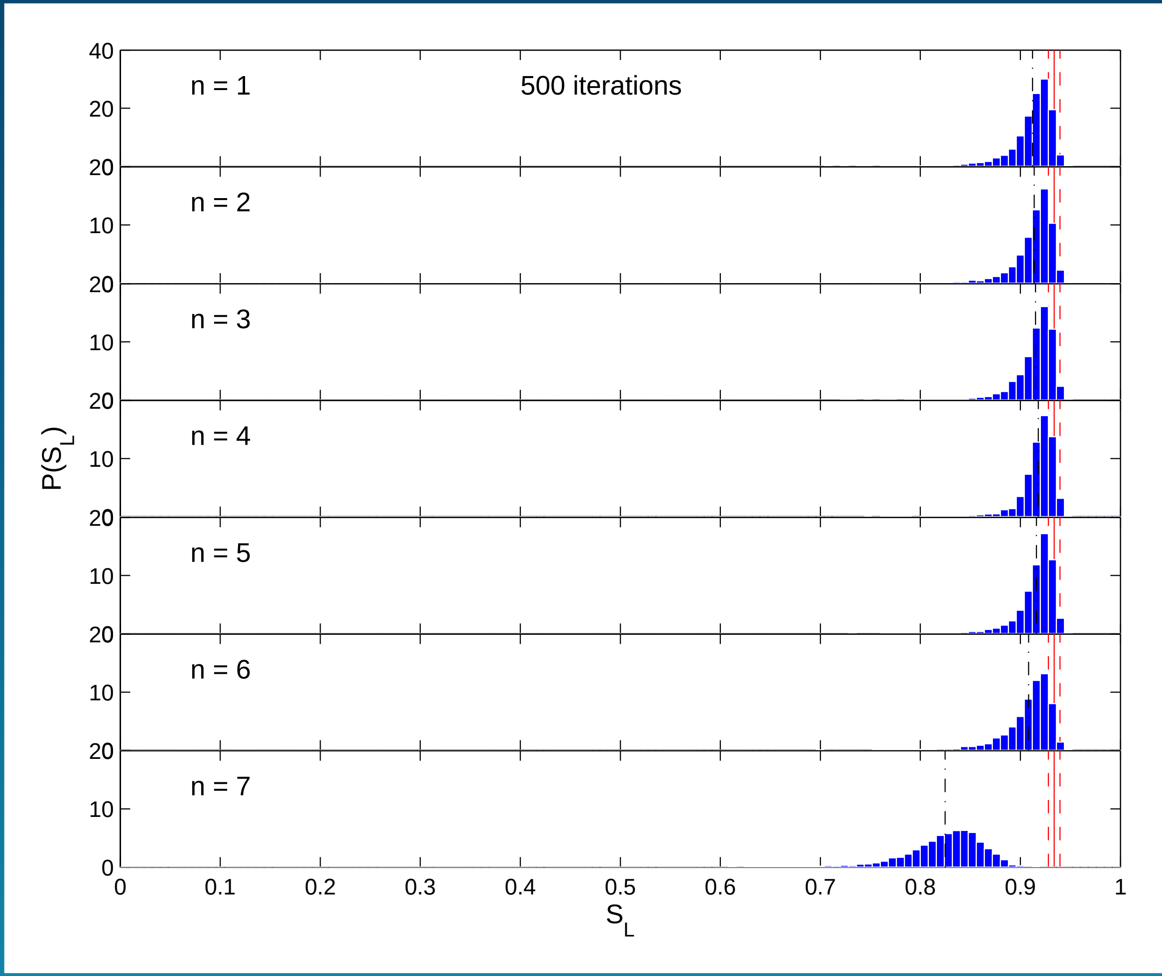

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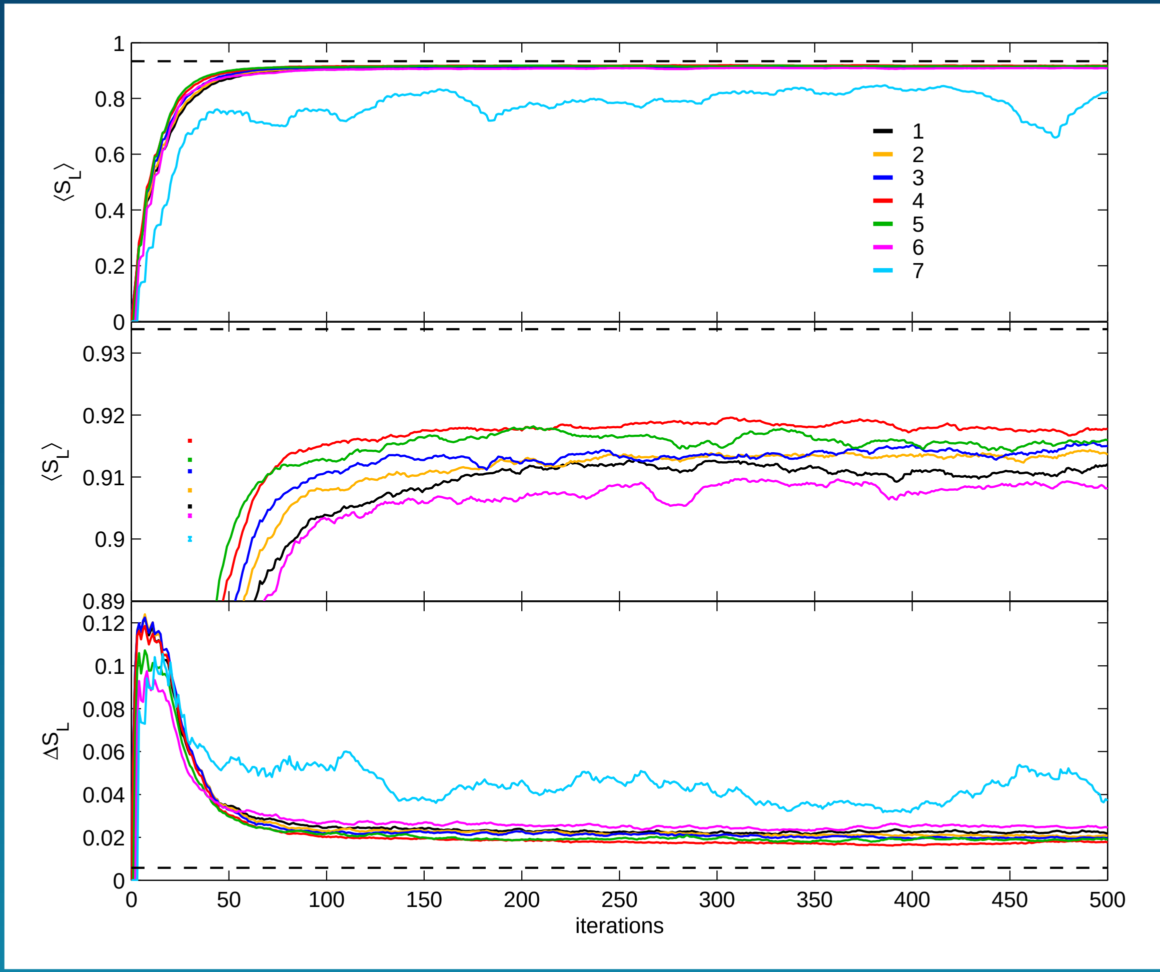


Entangling power

- We can also study pairwise (mixed-state) entanglement using the concurrence.

- The concurrence of a two-qubit density operator ρ is

$$C(\rho) \equiv \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the square roots of the eigenvalues of $\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. The complex conjugation is taken in the standard qubit basis.

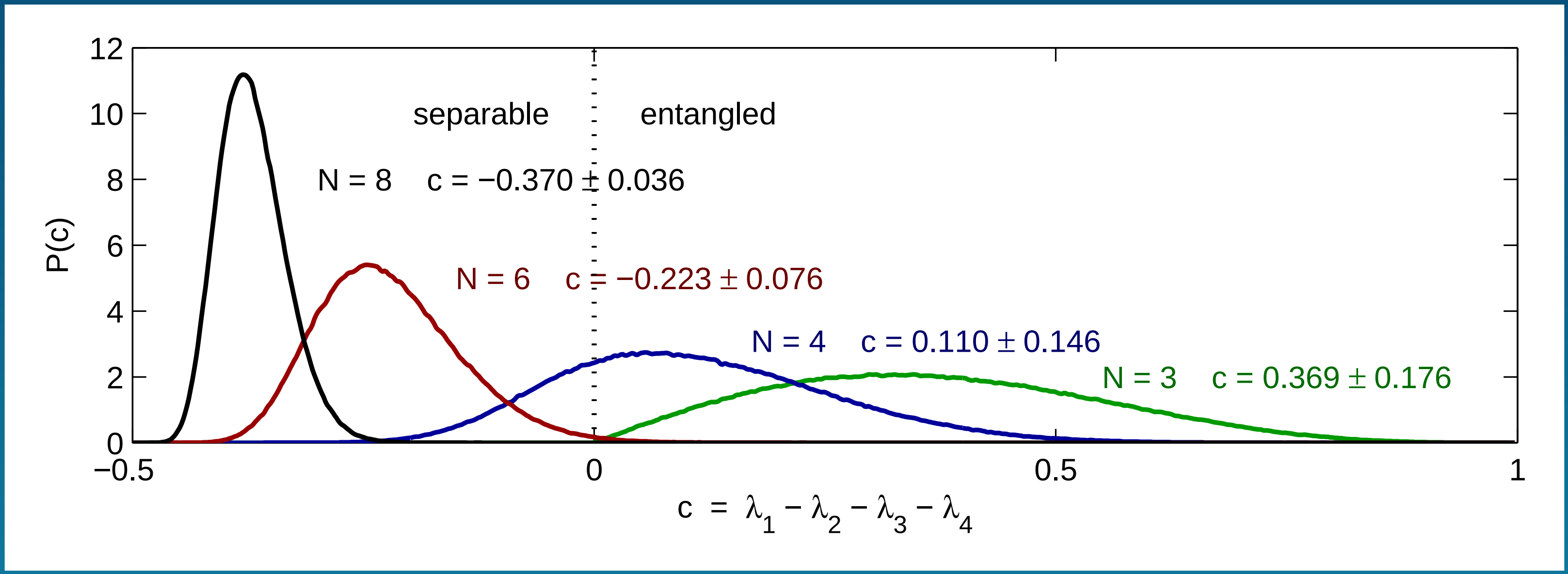
- It is more interesting to investigate the quantity

$$c(\rho) \equiv \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \qquad (-1/2 \leq c \leq 1)$$

Then $C(\rho) = \max\{0, c(\rho)\}$.

Entangling power

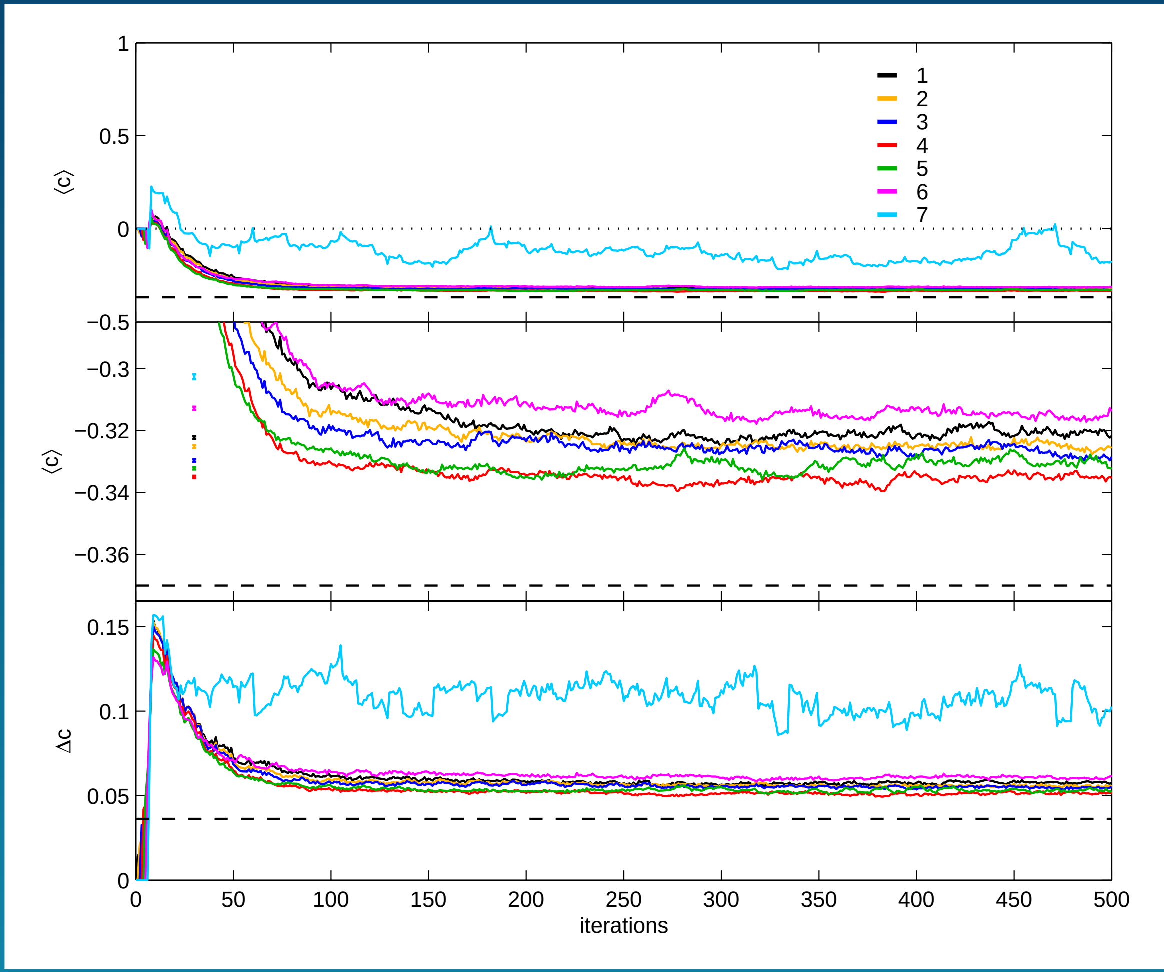
- We take ρ to be the density operator after tracing out all but two qubits of our N-qubit pure state. For random states:



- Typical multi-qubit pure states have LOW levels of pairwise entanglement.

Entangling power

- Baking entangled states with $N = 8$. The pairwise entanglement between the least significant and most significant qubit:



Entangling power

- A proposal for a measure of multipartite entanglement is the Meyer-Wallach measure Q which can be written as the average subsystem linear entropy of the constituent qubits:

$$Q(\psi) = 2\left(1 - \frac{1}{N}\sum_{k=1}^{N} \mathrm{tr}\,\rho_k^2\right)$$

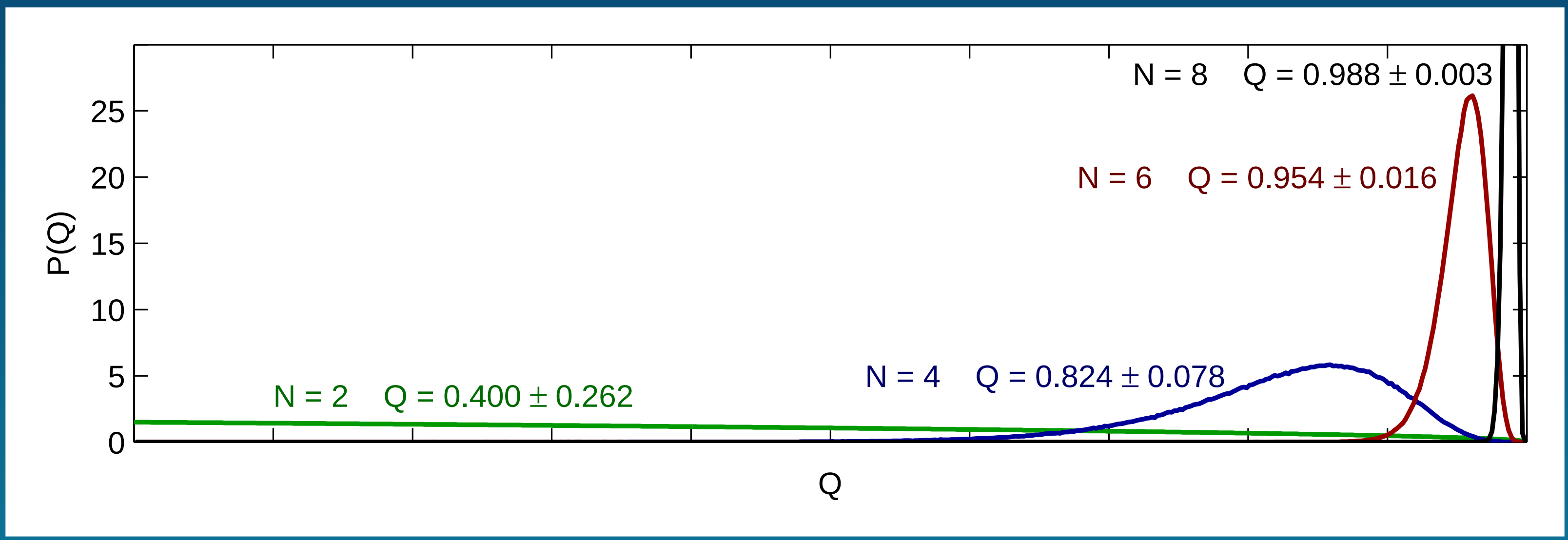
where ρ_k is the density operator for the k-th qubit after tracing out the rest.

- The mean and variance are

$$\begin{aligned} \langle Q \rangle &= \frac{D-2}{D+1} \\ \langle Q^2 \rangle - \langle Q \rangle^2 &= \frac{6(D-4)}{(D+3)(D+2)(D+1)N} + \frac{18D}{(D+3)(D+2)(D+1)^2} \end{aligned}$$

Entangling power

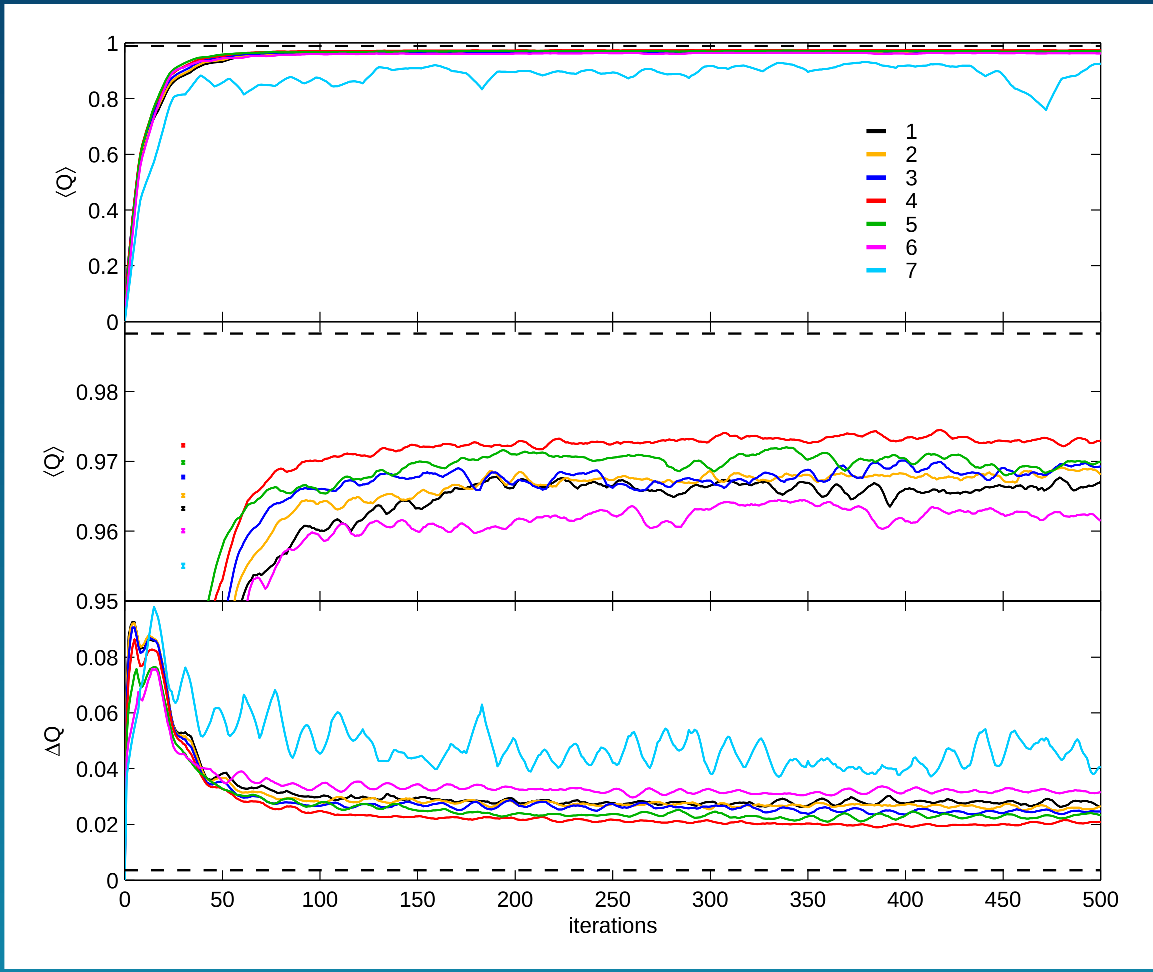
- For N-qubit random states:



- Typical multi-qubit pure states have HIGH levels of multipartite entanglement according to the measure Q.

Entangling power

- Baking entangled states with $N = 8$. The multipartite entanglement according to the measure Q:



Entangling power

Conclusion

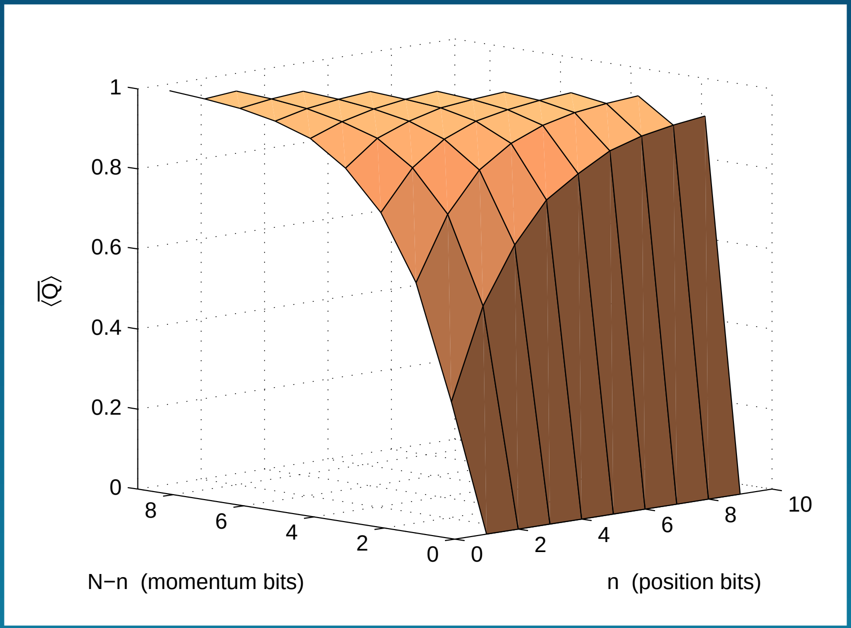
The quantum baker's maps are, for the most part, good at generating entanglement, producing multipartite entanglement close to that expected in random states.

We should expect high levels of entanglement creation in quantum maps that are chaotic in their classical limit. Such maps have a dynamical behavior that produces correlations between the coarse and fine scales of phase space. This behavior is described classically in the form of symbolic dynamics. Investigations into entanglement production, using the above qubit bases, allow us to characterize the quantum version of such correlations.

Relationship between the Entangling power and Classical limit?

- Define the long time saturation value of Q:

$$\overline{\langle Q \rangle} \equiv \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \langle Q \rangle_k$$



- No clear relationship between the entangling power and spurious classical limits.

Summary

- The Schack-Caves quantum baker's maps have the proper classical limit, provided the number of momentum bits is allowed to increase in this limit.

- The quantum baker's maps are, for the most part, good at generating entanglement, producing multipartite entanglement close to that expected in random states.

- Future directions might include exploring
  - ▷ multipartite entanglement production in regular systems via the qubit bases
  - ▷ tests of quantum chaos as applied to the quantum baker's maps, paying particular attention to the nonentangling trivial quantum baker's map