15 Duality for SUSY QCD

15.1 The Classical Moduli Space for $F \geq N$

	$SU(N)$	$SU(F)$	$SU(F)$	$U(1)$	$U(1)_R$
Q	$\square$	$\square$	$\mathbf{1}$	1	$\frac{F-N}{F}$
$\overline{Q}$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$	-1	$\frac{F-N}{F}$

Recall

$$D^a = g(\Phi^{*in}(T^a)^m_n \Phi_{mi} - \overline{\Phi}^{in}(T^a)^m_n \overline{\Phi}^*_{mi}) \tag{15.1}$$

and the potential is:

$$V = \frac{1}{2} D^a D^a \; . \tag{15.2}$$

Define

$$D^n_m \equiv \langle \Phi^{*in} \Phi_{mi} \rangle \tag{15.3}$$

$$\overline{D}^n_m = \langle \overline{\Phi}^{in} \overline{\Phi}^*_{mi} \rangle \tag{15.4}$$

D^n_m and $\overline{D}^n_m$ are $N \times N$ positive semi-definite Hermitian matrices. In a vacuum state we must have:

$$D^a = g T^{am}_n (D^n_m - \overline{D}^n_m) = 0 \tag{15.5}$$

Since T^a is a complete basis for traceless matrices, we must have

$$D^n_m - \overline{D}^n_m = \rho I \tag{15.6}$$

D^n_m can be diagonalized by an $SU(N)$ gauge transformation

$$U^\dagger D U \tag{15.7}$$

$$D = \begin{pmatrix} |v_1|^2 & & & \\ & |v_2|^2 & & \\ & & \ddots & \\ & & & |v_N|^2 \end{pmatrix} \tag{15.8}$$

In this basis $\overline{D}^n_m$ must also be diagonal, with eigenvalues $|\overline{v}_i|^2$. This tells us that

$$|v_i|^2 = |\overline{v}_i|^2 + \rho \tag{15.9}$$

D^n_m and $\overline{D}^n_m$ are invariant under flavor transformations, thus, up to flavor transformations, we can write

$$\langle\Phi\rangle = \begin{pmatrix} v_1 & & & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & v_N & 0 & \dots & 0 \end{pmatrix} \tag{15.10}$$

$$\langle\overline{\Phi}\rangle = \begin{pmatrix} \overline{v}_1 & & \\ & \ddots & \\ & & \overline{v}_N \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \tag{15.11}$$

At a generic point in the moduli space the $SU(N)$ gauge symmetry is broken completely and there are $2NF - 2(N^2 - 1)$ massless chiral supermultiplets. We can descibe these light degrees of freedom in a gauge invariant way by scalar "meson" and "baryon" fields and there superpartners:

$$\begin{aligned} M^j_i &= \overline{\Phi}^{j\alpha}\Phi_{\alpha i} && (15.12) \\ B_{i_1,\dots,i_N} &= \Phi_{\alpha_1 i_1}\dots\Phi_{\alpha_N i_N}\epsilon^{\alpha_1,\dots,\alpha_N} && (15.13) \\ \overline{B}^{i_1,\dots,i_N} &= \overline{\Phi}^{\alpha_1 i_1}\dots\overline{\Phi}^{\alpha_N i_N}\epsilon_{\alpha_1,\dots,\alpha_N} && (15.14) \\ & && (15.15) \end{aligned}$$

The fermion partners of these fields are products of scalars and fermions. Since the fields M, B and $\overline{B}$ have $2\begin{pmatrix} F \\ N \end{pmatrix} + F^2$ components there are constraints relating them, since they are constructed out of the same fields. For example

$$B_{i_1,\dots,i_N}\overline{B}^{j_1,\dots,j_N} = M^{j_1}_{[i_1}\dots M^{j_N}_{i_N]} \tag{15.16}$$

where [] denotes antisymmetrization.

Up to flavor transformations the moduli space is described by:

$$\langle M \rangle = \begin{pmatrix} v_1\overline{v}_1 & & & & & \\ & \ddots & & & & \\ & & v_N\overline{v}_N & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \tag{15.17}$$

$$\langle B_{1,\ldots,N} \rangle = v_1 \ldots v_N \tag{15.18}$$

$$\langle \overline{B}^{1,\ldots,N} \rangle = \overline{v}_1 \ldots \overline{v}_N \tag{15.19}$$

with all other components set to zero. The rank of M is at most N. If it is less than N, then B or $\overline{B}$ (or both) vanish. If the rank of M is k, then $SU(N)$ is broken to $SU(N-k)$ with $F-k$ flavors.

15.2 The Quantum Moduli Space for $F \geq N$

Recall that the ADS superpotential made no sense for $F \geq N$ however the vacuum solution

$$M_i^j = (m^{-1})_i^j \left(\text{det}m\Lambda^{3N-F}\right)^{\frac{1}{N}} \tag{15.20}$$

is still sensible. Giving large masses to flavors N through F and matching the gauge coupling at the mass thresholds gives

$$\Lambda^{3N-F}\text{det}m_H = \Lambda_{N,N-1}^{2N+1} \tag{15.21}$$

The low-energy effective theory has $N-1$ flavors and an ADS superpotential. If we give a small mass to the light flavors we have

$$\begin{aligned} M_i^j &= (m_L^{-1})_i^j \left(\text{det}m_L\Lambda_{N,N-1}^{2N+1}\right)^{\frac{1}{N}} \\ &= (m_L^{-1})_i^j \left(\text{det}m_L\text{det}m_H\Lambda^{3N-F}\right)^{\frac{1}{N}} \end{aligned} \tag{15.22}$$

Since the masses are holomorphic parameters of the theory, this relationship can only break down at isolated points, so eq. (15.20) is true in general. For $F \geq N$ we can take $m_j^i \to 0$ with components of M finite or zero. So the vacuum degeneracy is not lifted and there is a quantum moduli space for $F \geq N$. However the classical constraints between M, B and $\overline{B}$ may be modified.

15.3 Duality

For $F \geq 3N$ we lose asymptotic freedom, so the theory can be understood as a weakly coupled low-energy effective theory. For F just below $3N$ we have an infrared fixed point. Recall

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{(3N - F(1-\gamma))}{1 - N\frac{g^2}{8\pi^2}} \tag{15.23}$$

where

$$\gamma = -\frac{g^2}{8\pi^2}\frac{N^2-1}{N} + \mathcal{O}(g^4) \tag{15.24}$$

So

$$\begin{aligned} 16\pi^2\beta(g) &= -g^3\left(3N - F\right) - \frac{g^5}{8\pi^2}\left(3N^2 - FN - F\frac{N^2-1}{N}\right) \\ &+\mathcal{O}(g^7) \end{aligned} \tag{15.25}$$

For $F = 3N - \epsilon N$

$$16\pi^2\beta(g) = -g^3\epsilon N - \frac{g^5}{8\pi^2}\Big(3(N^2-1) + \mathcal{O}(\epsilon)\Big) \tag{15.26}$$

So there is a IR fixed point at

$$g_*^2 = \frac{8\pi^2}{3}\frac{N}{N^2-1}\epsilon \tag{15.27}$$

A general result is that a scale invariant theory of fields with spin ≤ 1 is actually conformally invariant. In a conformal SUSY theory the SUSY algebra is extended to a superconformal algebra. A particular R charge enters the algebra in an important way. One finds that the dimensions of fields are bounded

$$D \geq \frac{3}{2}|R_{\rm sc}| \tag{15.28}$$

and the inequality is saturated for chiral and anti-chiral fields:

$$D = \frac{3}{2}R_{\rm sc}, \ \text{for chiral fields} \tag{15.29}$$

$$D = -\frac{3}{2}R_{\rm sc}, \ \text{for anti} - \text{chiral fields} \tag{15.30}$$

Since

$$R_{\rm sc}(\mathcal{O}_1\mathcal{O}_2) = R_{\rm sc}(\mathcal{O}_1) + R_{\rm sc}(\mathcal{O}_2) \tag{15.31}$$

we have for chiral fields

$$D(\mathcal{O}_1\mathcal{O}_2) = D(\mathcal{O}_1) + D(\mathcal{O}_2) \tag{15.32}$$

In general $R_{\rm sc}$ is ambiguous, since we can form linear combinations of $U(1)$'s, but for SUSY QCD $R_{\rm sc}$ is unique since we must have

$$R_{\rm sc}(Q) = R_{\rm sc}(\overline{Q}) \tag{15.33}$$

so we can identify the R charge we have been using with $R_{\rm sc}$. So at the IR fixed point

$$\begin{aligned} D(M) = D(\Phi\overline{\Phi}) = 2 + \gamma_* &= \frac{3}{2}2\frac{(F-N)}{F} \\ &= 3 - \frac{3N}{F} \end{aligned} \tag{15.34}$$

We can check that the exact β function vanishes:

$$\beta \propto 3N - F + F\gamma_* = 0 \tag{15.35}$$

For a scalar field in a conformal theory we also have

$$D(\phi) \geq 1 \tag{15.36}$$

with equality for a free field. Requiring that $D(Q\overline{Q}) \geq 1$ implies

$$F \geq \frac{3}{2}N \tag{15.37}$$

In a conformal theory (even if it is strongly coupled) we don't expect any global symmetries to break, so 't Hooft anomaly matching should apply to any description of the low-energy degrees of freedom.

Seiberg found a solution to the anomaly matching:

	$SU(F-N)$	$SU(F)$	$SU(F)$	$U(1)$	$U(1)_R$
q	$\Box$	$\overline{\Box}$	$\mathbf{1}$	$\frac{N}{F-N}$	$\frac{N}{F}$
$\overline{q}$	$\overline{\Box}$	$\mathbf{1}$	$\Box$	$-\frac{N}{F-N}$	$\frac{N}{F}$
$\widetilde{M}$	$\mathbf{1}$	$\Box$	$\overline{\Box}$	0	$2\frac{F-N}{F}$

The anomalies match as follows:

$$\begin{aligned}
&SU(F)^3 : -(F-N) + F = N \\
&U(1)SU(F)^2 : \frac{N}{F-N}(F-N)\frac{1}{2} = \frac{N}{2} \\
&U(1)_R SU(F)^2 : \frac{N-F}{F}(F-N)\frac{1}{2} + \frac{F-2N}{F}F\frac{1}{2} = -\frac{N^2}{2F} \\
&U(1)^3 : 0 \\
&U(1) : 0 \\
&U(1)U(1)_R^2 : 0 \\
&U(1)_R : \left(\frac{N-F}{F}\right)2(F-N)F + \left(\frac{F-2N}{F}\right)F^2 + (F-N)^2 - 1 \\
&\qquad = -N^2 - 1 \\
&U(1)_R^3 : \left(\frac{N-F}{F}\right)^3 2(F-N)F + \left(\frac{F-2N}{F}\right)^3 F^2 + (F-N)^2 - 1 \\
&\qquad = -\frac{2N^4}{F^2} + N^2 - 1 \\
&U(1)^2U(1)_R : \left(\frac{N}{F-N}\right)^2 \frac{N-F}{F} 2F(F-N) = -2N^2
\end{aligned} \tag{15.38}$$

This theory admits a unique superpotential:

$$W = \lambda \widetilde{M}_i^j \phi_j \overline{\phi}^i \tag{15.39}$$

This ensures that the two theories have the same number of degrees of freedom since the $\widetilde{M}$ eq. of motion removes the color singlet $\phi\overline{\phi}$ degrees of freedom. The dual theory also has baryon operators:

$$b^{i_1,\ldots,i_{F-N}} = \phi^{\alpha_1 i_1} \ldots \phi^{\alpha_{F-N} i_{F-N}} \epsilon_{\alpha_1,\ldots,\alpha_{F-N}} \tag{15.40}$$

$$\overline{b}_{i_1,\ldots,i_{F-N}} = \overline{\phi}_{\alpha_1 i_1} \ldots \overline{\phi}_{\alpha_{F-N} i_{F-N}} \epsilon^{\alpha_1,\ldots,\alpha_{F-N}} \tag{15.41}$$

$$\tag{15.42}$$

The two moduli spaces have a mapping

$$\begin{aligned}
&M \leftrightarrow \widetilde{M} \\
&B_{i_1,\ldots,i_N} \leftrightarrow \epsilon_{i_1,\ldots,i_N,j_1,\ldots j_{F-N}} b^{j_1,\ldots,j_{F_N}} \\
&\overline{B}^{i_1,\ldots,i_N} \leftrightarrow \epsilon^{i_1,\ldots,i_N,j_1,\ldots j_{F-N}} \overline{b}_{j_1,\ldots,j_{F_N}}
\end{aligned} \tag{15.43}$$

The counting works because $2FN - 2(N^2-1) = 2F\widetilde{N} - 2(\widetilde{N}^2-1)$ where $\widetilde{N} = F - N$ is the number of colors in the dual theory. The one-loop β function in the dual theory is

$$\beta(\widetilde{g}) \propto -\widetilde{g}^3(3\widetilde{N} - F) = -\widetilde{g}^3(2F - 3N) \tag{15.44}$$

So the dual theory loses asymptotic freedom when $F \leq 3N/2$. When

$$F = 3\widetilde{N} - \epsilon\widetilde{N} \tag{15.45}$$

there is a perturbative fixed point at

$$\widetilde{g}_*^2 = \frac{8\pi^2}{3}\frac{\widetilde{N}}{\widetilde{N}^2-1}\left(1 + \frac{F}{\widetilde{N}}\right)\epsilon \tag{15.46}$$

$$\lambda_*^2 = \frac{16\pi^2}{3\widetilde{N}}\epsilon \tag{15.47}$$

At this fixed point $D(\widetilde{M}\overline{\phi}\phi) = 3$, so the superpotential term is marginal.

At $\lambda = 0$ $\widetilde{M}$ has no interactions so its dimension is 1. We can calculate the dimension of $\phi\overline{\phi}$ from the R charge for $F > 3N/2$:

$$D(\phi\overline{\phi}) = \frac{3(F - \widetilde{N})}{F} = \frac{3N}{F} < 2 \tag{15.48}$$

So the superpotential is a relevant operator (not marginal) and there is an unstable fixed point for

$$\widetilde{g}^2 = \frac{8\pi^2}{3}\frac{\widetilde{N}}{\widetilde{N}^2-1}\epsilon \tag{15.49}$$

$$\lambda^2 = 0 \tag{15.50}$$

So we have found that SUSY QCD has an interacting IR fixed point for $3N/2 < F < 3N$. Such conformal theories have no particle interpretation, but anomalous dimensions are physical quantities.

For $N + 1 < F \leq 3N/2$ the IR fixed point of the dual theory is trivial (aymptotic freedom is lost in the dual):

$$\widetilde{g}_*^2 = 0 \tag{15.51}$$

$$\lambda_*^2 = 0 \tag{15.52}$$

Since $\widetilde{M}$ has no interactions it has dimension 1, and there is an accidental $U(1)$ symmetry. For this range of F, $R_{\rm sc}$ is a linear combination of R and this accidental $U(1)$. This is consistent with the relation $D(\widetilde{M}) = (3/2)R_{\rm sc}(\widetilde{M})$. Surprisingly in this range we find in the IR free massless composite gauge bosons, quarks, mesons, and their superpartners.

What we have found is two different theories that have infrared fixed points that describe the same physics. This is just another example of finding two theories which are in the same universality class. A well known example of this is QCD and the chiral Lagrangian. This is a useful thing to do if one theory is strongly coupled and the other is weakly coupled. (TWo different theories could not both be weakly coupled and describe the same physics.) Here we see that even when the dual theory cannot be thought of as being composites of the original degrees of freedom it provides a weakly coupled description in the region where the original theory is strongly coupled. The name duality is tacked-on because both theories happen to be gauge theories.

It is common (though confusing) to write

$$\lambda\widetilde{M} = \frac{M}{\mu} \tag{15.53}$$

which trades λ for a scale μ and uses the same symbol, M, for fields in the two different theories.

15.4 Integrating out a flavor

If we give a mass to one flavor in the original theory we have a superpotential

$$W = m\overline{\Phi}^F \Phi_F \tag{15.54}$$

In the dual theory we have

$$W_d = \frac{1}{\mu} M_i^j \overline{\phi}^i \phi_j + m M_F^F \tag{15.55}$$

The eq. of motion for M_F^F is

$$\frac{\delta W_d}{\delta M_F^F} = \frac{1}{\mu}\overline{\phi}^F \phi_F + m \tag{15.56}$$

So $\overline{\phi}^F \phi_F = -\mu m$. We saw that along such a D flat direction we have a theory with one less color, one less flavor, and some singlets. The spectrum is:

	$SU(F-N-1)$	$SU(F-1)$	$SU(F-1)$
q'	$\square$	$\overline{\square}$	$\mathbf{1}$
$\overline{q}'$	$\overline{\square}$	$\mathbf{1}$	$\square$
M'	$\mathbf{1}$	$\square$	$\overline{\square}$
q''	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$
$\overline{q}''$	$\mathbf{1}$	$\mathbf{1}$	$\square$
S	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
M_j^F	$\mathbf{1}$	$\square$	$\mathbf{1}$
M_F^j	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square}$
M_F^F	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

The superpotential is

$$W_{\text{eff}} = \frac{v}{\mu}\left(M_F^j \phi_j'' + M_i^F \overline{\phi}''^i + M_F^F S\right) + M'\overline{\phi}'\phi' \tag{15.57}$$

So we can integrate out M_F^j, ϕ_j'', M_i^F, $\overline{\phi}''^i$, M_F^F, and S. This leaves just the dual of $SU(N)$ with $F-1$ flavors.

15.5 Consistency

We have seen that the conjectured duality satisfies three non-trivial consistency checks:

- The global anomalies of the original quarks and gauginos match those of the dual quarks, dual gauginos, and "mesons".
- The moduli spaces have the same dimenions and the gauge invariant operators match:

$$\begin{aligned} &M \leftrightarrow \widetilde{M} \\ &B_{i_1,\ldots,i_N} \leftrightarrow \epsilon_{i_1,\ldots,i_N,j_1,\ldots j_{F-N}} b^{j_1,\ldots,j_{F_N}} \\ &\overline{B}^{i_1,\ldots,i_N} \leftrightarrow \epsilon^{i_1,\ldots,i_N,j_1,\ldots j_{F-N}} \overline{b}_{j_1,\ldots,j_{F_N}} \end{aligned} \tag{15.58}$$

- Integrating out a flavor in the original theory results in an $SU(N)$ theory with $F-1$ flavors, which should have a dual with $SU(F-N-1)$ and $F-1$ flavors. Starting with the dual of the original theory, the mapping of the mass term is a linear term for the "meson" which forces the dual squarks to have a VEV and Higgses the theory down to $SU(F-N-1)$ and $F-1$ flavors.

The duality exchanges weak and strong coupling and also classical and quantum effects. For example in the original theory M satisfies a classical constaint $\mathrm{rank}(M) \leq N$. In the dual theory there are $F - \mathrm{rank}(M)$ light dual quarks. If $\mathrm{rank}(M) > N$ then the number of light dual quarks is less than $\widetilde{N} = F - N$, and an ADS superpotential is generated, so there is no vacuum. Thus in the dual, $\mathrm{rank}(M) \leq N$ is enforced by quantum effects.

References

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[2] A. de Gouvea, A. Friedland and H. Murayama, "Seiberg duality and e+ e- experiments," Phys. Rev. **D59** (1999) 105008 hep-th/9810020.